

On Strong Embeddings by Stein's Method

Chinmoy Bhattacharjee and Larry Goldstein

Department of Mathematics, University of Southern California

Abstract

Strong embeddings, that is, couplings between a partial sum process of a sequence of random variables and a Brownian motion, have found numerous applications in probability and statistics. We extend Chatterjee's novel use of Stein's method for $\{-1, +1\}$ valued variables to a general class of discrete distributions, and provide $\log n$ rates for the coupling of partial sums of independent variables to a Brownian motion, and results for coupling sums of suitably standardized exchangeable variables to a Brownian bridge.

1 Introduction

Let $\varepsilon_1, \varepsilon_2 \dots$ be a sequence of independent random variables distributed as ε , a mean zero, variance one random variable. Letting $S_k = \sum_{i=1}^k \varepsilon_i, k = 1, 2, \dots$, be the corresponding sequence of partial sums, Donsker's invariance principle [11], see also [3], implies that the random continuous function

$$X_n(t) = \frac{1}{\sqrt{n}}(S_{[nt]} + (nt - [nt])\varepsilon_{[nt]+1}), \quad 0 \leq t \leq 1$$

converges weakly to a Brownian motion process $(B_t)_{0 \leq t \leq 1}$. One way to study the quality of the approximation of $X_n(t)$ by B_t is to determine a 'slowly increasing' sequence $f(n)$ such that there exists an embedding of both processes on a common probability space such that

$$\max_{0 \leq k \leq n} |S_k - B_k| = O_p(f(n)).$$

Finding the smallest achievable order of $f(n)$ has been a very important question in the literature.

The rate $(n \log \log n)^{1/4} (\log n)^{1/2}$ was achieved by Skorokhod [23], also see its translation [24] and Strassen [27] assuming $\mathbb{E}\varepsilon^4 < \infty$ using Skorokhod embedding, and Kiefer [16] showed that this rate was optimal under the finite fourth moment condition. Csörgő and Révész [8] made improvements to the rate under additional moment assumptions. See the survey paper by Oblój [20] and [9] for a more detailed account.

The celebrated KMT approximation by Komlós, Major and Tusnády ([17], [18]) achieved the rate $\log n$ under the condition that ε have a finite moment generating function in a neighborhood of zero. To state their result precisely we make the following definition.

We say Strong Embedding (SE) holds for the mean zero, variance one random variable ε if there exist constants C, K , and λ such that for all $n = 1, 2, \dots$ the partial sums $S_k = \sum_{i=1}^k \varepsilon_i, k = 1, \dots, n$ of a

sequence $\varepsilon_1, \varepsilon_2 \dots$ of independent random variables distributed as ε , and a standard Brownian motion $(B_t)_{t \geq 0}$ can be constructed on a joint probability space such that

$$P\left(\max_{0 \leq k \leq n} |S_k - B_k| \geq C \log n + x\right) \leq K e^{-\lambda x} \quad \text{for all } x \geq 0. \quad (1)$$

We adopt the standard empty sum convention whereby $S_0 = 0$.

Theorem 1.1 (KMT approximation [17]). *SE holds for ε satisfying $\mathbb{E} \exp \theta |\varepsilon| < \infty$ for some $\theta > 0$.*

Results by Bártfai [1], see [31], show that the rate in (1) is best possible under the finite moment generating function condition. A multidimensional version of the KMT approximation was proved by Einmahl [12], from which Zaitsev ([29], [30]) removed a logarithmic factor. For extensions to stationary sequences see the history in [2], where dependent variables of the form $X_k = G(\dots, \varepsilon_{k-1}, \varepsilon_k, \varepsilon_{k+1}, \dots)$ for $\varepsilon_i, i \in \mathbb{Z}$ i.i.d. are considered. Strong embedding results have a truly extensive range of applications that includes empirical processes, non-parametric statistics, survival analysis, time series, and reliability; for a sampling see the texts [22] [9], or the articles [10], [28] and [21].

Here we take the approach to the KMT approximation introduced by Chatterjee [6] that has its origins in Stein's method [26] and appears simpler, and is possibly easier to generalize, than the dyadic approximation argument of [17]. This alternative approach depends on the use of Stein coefficients, also known as Stein kernels, that first appeared in the work of Cacoullos and Papathanasiou [5]. In some sense, a Stein coefficient T for a mean zero random variable W neatly encodes all information regarding the closeness of W to the mean zero normal variable Z having variance σ^2 . Theorem 2.3 below, from [6], demonstrates that a coupling of W and Z exists whose quality can be evaluated uniquely as a function of T and σ^2 . Theorem 1.2, that demonstrates Theorem 1.1 for the special case of simple symmetric random walk, was proved in [6] applying this approach.

Theorem 1.2 (Chatterjee [6]). *SE holds for ε a symmetric random variable with support $\{-1, +1\}$.*

In this work, using the methods of [6], we generalize Theorem 1.2 as follows.

Theorem 1.3. *SE holds for ε , any random variable with mean zero and variance 1 satisfying $\mathbb{E} \varepsilon^3 = 0$, taking values in a finite set \mathcal{A} not containing 0.*

To prove our result we first provide a construction in the case where we have a finite number of variables and then extend to derive strong approximation for an infinite sequence. Such extensions have been studied in the context of the KMT theorem for summands with finite p -th moment in [19] and also in [6].

For the finite case we employ induction, as in [6]. The induction step requires extending Theorem 1.4 of [6] from the special case where ε is a symmetric variable taking values in $\{-1, 1\}$. The generalization depends on the ‘zero-bias’ smoothing method introduced in Lemma 2.4, which may be of independent interest as regards the construction of Stein coefficients. Theorem 1.4 here is a new result for the embedding of exchangeable random variables and a Brownian bridge.

Theorem 1.4. *For any positive integer n , let $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n$ be exchangeable random variables taking values in a finite set $\mathcal{A} \subset \mathbb{R}$. Let*

$$S_k = \sum_{i=1}^k \varepsilon_i, \quad W_k = S_k - \frac{k}{n} S_n \quad \text{and} \quad \gamma^2 = \frac{1}{n} \sum_{i=1}^n \varepsilon_i^2.$$

Then there exists a positive universal constant C , and for all $v > 0$ positive constants K_1, K_2 and λ_0 depending only on \mathcal{A} and v , such that for all $n \geq 1$ and $\eta \geq v$, a version of W_0, W_1, \dots, W_n and a standard Brownian bridge $(B_t)_{0 \leq t \leq 1}$ exist on the same probability space and satisfy

$$\begin{aligned} \mathbb{E} \exp(\lambda \max_{0 \leq k \leq n} |W_k - \sqrt{n} \eta B_{k/n}|) \\ \leq \exp(C \log n) \mathbb{E} \exp\left(\frac{K_1 \lambda^2 S_n^2}{n} + K_2 \lambda^2 n (\gamma^2 - \eta^2)^2\right) \text{ for all } \lambda \leq \lambda_0. \end{aligned}$$

Moreover, if $0 \notin \mathcal{A}$, then there exist positive constants K_1 and λ_0 depending only on \mathcal{A} such that

$$\mathbb{E} \exp(\lambda \max_{0 \leq k \leq n} |W_k - \sqrt{n} \gamma B_{k/n}|) \leq \exp(C \log n) \mathbb{E} \exp\left(\frac{K_1 \lambda^2 S_n^2}{n}\right) \text{ for all } \lambda \leq \lambda_0,$$

and if in addition $\varepsilon_1, \dots, \varepsilon_n$ are i.i.d. with zero mean, then there exists a positive λ depending only on \mathcal{A} such that

$$P\left(\max_{0 \leq k \leq n} |W_k - \sqrt{n} \gamma B_{k/n}| \geq \lambda^{-1} C \log n + x\right) \leq 2e^{-\lambda x} \text{ for all } x \geq 0.$$

The constant C is given explicitly in (41) in the proof of Theorem 3.1; its numerical value is roughly 8.4. The constants in the second inequality of Theorem 1.4 are those that appear in the first inequality, specialized to a case where the lower bound v depends only on \mathcal{A} .

Our extension of the Rademacher variable result of [6] requires a number of non-trivial components. Example 3 of [6] demonstrates how to smooth Rademacher variables to obtain Stein coefficients, and the author states ‘we do not know yet how to use Theorem 1.2 to prove the KMT theorem in its full generality, because we do not know how to generalize the smoothing technique of Example 3.’ We address this point by the zero bias method of Lemma 2.4, that shows how any mean zero, finite variance random variable may be smoothed to obtain a Stein coefficient.

Additionally, dealing with variables restricted to the set $\{-1, 1\}$ avoids another difficulty. In particular, the second inequality of Theorem 1.4 shows that the ‘natural scaling’ for the approximating Brownian bridge process depends on the variance parameter $\gamma^2 = n^{-1} \sum_{i=1}^n \varepsilon_i^2$, which in the case of Rademacher variables is always one. In fact, for such variables, the variance parameter remains the constant one when restricted and suitably scaled to any subset of variables. In contrast, in general when applying induction to piece together a larger path from smaller ones, their respective variance parameters may not match. This effect gives rise to the term $(\gamma^2 - \eta^2)^2$ in the exponent of the first inequality of Theorem 1.4, which then needs to be controlled in order for the induction to be completed. In doing so, one gains results on the comparison of the sample paths of a more general classes of exchangeable variables to a Brownian bridge.

The second claim of Theorem 1.4 is shown under the assumption $0 \notin \mathcal{A}$. This condition becomes critical precisely at (63), where we require that the smallest absolute value of the elements of \mathcal{A} is positive, from which one then obtains a lower bound v on γ when invoking Theorem 3.1. This same phenomenon occurs in the proof of Lemma 4.1, on the way to demonstrate Theorem 1.3.

The remainder of this work is organized as follows. In Section 2, we prove two theorems, one for coupling sums S_n of i.i.d. random variables, and one for coupling W_n of Theorem 1.4, to Gaussians. We also prove Lemma 2.4, which shows how to construct Stein type coefficients using smoothing by zero bias variables. Theorems 3.1 and 1.4, the first result a conditional version of the second, are proved in Section 3, and we prove Lemma 4.1, implying Theorem 1.3, in Section 4.

2 Bounds for couplings to Gaussian variables

In this section we prove Theorems 2.1 and 2.2, generalizations of Theorems 3.1 and 3.2 of [6], and our zero bias smoothing result, Lemma 2.4. The first theorem gives bounds on couplings of sums S_n of i.i.d. variables, and the second on coupling of certain exchangeable sums to Gaussian random variables.

Theorem 2.1. *For every mean zero, variance one bounded random variable ε satisfying $\mathbb{E}(\varepsilon^3) = 0$ and $\mathbb{E}(\varepsilon^4) < \infty$, there exists $\theta_1 > 0$ such that for every positive integer n it is possible to construct a version of the sum $S_n = \sum_{i=1}^n \varepsilon_i$ of n independent copies of ε , and $Z_n \sim \mathcal{N}(0, n)$, on a joint probability space such that*

$$\mathbb{E} \exp(\theta_1 |S_n - Z_n|) \leq 8.$$

For convenience, we adopt the convention that a normal random variable with mean μ and zero variance is identically equal to μ .

Theorem 2.2. *For $n \geq 1$, let $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n$ be arbitrary elements of a finite set $\mathcal{A} \subset \mathbb{R}$, not necessarily distinct. Let $\gamma^2 = n^{-1} \sum_{i=1}^n \varepsilon_i^2$, let π be a uniform random permutation of $\{1, 2, \dots, n\}$, and for each $1 \leq k \leq n$ let*

$$S_k = \sum_{i=1}^k \varepsilon_{\pi(i)} \quad \text{and} \quad W_k = S_k - \frac{kS_n}{n}. \quad (2)$$

Then for all $\nu > 0$ there exist positive constants c_1, c_2 and θ_2 depending only on \mathcal{A} and ν such that for any integer $n \geq 1$, an integer k such that $|2k - n| \leq 1$, and any $\eta \geq \nu$, it is possible to construct a version of W_k and a Gaussian random variable Z_k with mean 0 and variance $k(n-k)/n$ on the same probability space such that for all $\theta \leq \theta_2$,

$$\mathbb{E} \exp(\theta |W_k - \eta Z_k|) \leq \exp \left(3 + \frac{c_1 \theta^2 S_n^2}{n} + c_2 \theta^2 n (\gamma^2 - \eta^2)^2 \right).$$

We now define Stein coefficients, the key ingredient upon which our approach depends. Let W be a random variable with $\mathbb{E}[W] = 0$ and finite second moment. We say the random variable T defined on the same probability space is a Stein coefficient for W if

$$\mathbb{E}[W f(W)] = \mathbb{E}[T f'(W)] \quad (3)$$

for all Lipschitz functions f and f' any a.e. derivative of f , whenever these expectations exist.

Theorem 2.3 (Chatterjee [6]). *Let W be mean zero with finite second moment and suppose that T is a Stein coefficient for W with $|T|$ almost surely bounded by a constant. Then, given any $\sigma^2 > 0$, we can construct a version of W and $Z \sim \mathcal{N}(0, \sigma^2)$ on the same probability space such that*

$$\mathbb{E} \exp(\theta |W - Z|) \leq 2 \mathbb{E} \exp \left(\frac{2\theta^2 (T - \sigma^2)^2}{\sigma^2} \right) \quad \text{for all } \theta \in \mathbb{R}.$$

To prove Theorems 2.1 and 2.2, we require the following definitions. Following Section 3.2 of [13], see also Proposition 4.2 of [7], for X a random variable with finite, non-zero second moment, we say X^\square has the X -square bias distribution when

$$\mathbb{E}[f(X^\square)] = \frac{1}{\mathbb{E}X^2} \mathbb{E}[X^2 f(X)] \quad (4)$$

for all functions f for which the expectation on the right hand side exists. For a mean zero random variable X with finite, non-zero variance σ^2 , we say that X^* has the X -zero bias distribution when

$$\sigma^2 \mathbb{E}[f'(X^*)] = \mathbb{E}[X f(X)] \quad (5)$$

for all Lipschitz functions f and any a.e. derivative f' , whenever these expectations exist. That X^* exists for such random variables, see [14] and [7].

If X is a mean zero random variable with finite, non-zero variance σ^2 , then for any $g \in C_c$, the collection of continuous functions with compact support, letting $f(x) = \int_0^x g(u) du$, using (4), we have

$$\begin{aligned} \sigma^2 \mathbb{E}g(UX^\square) &= \sigma^2 \mathbb{E}f'(UX^\square) \\ &= \sigma^2 \mathbb{E} \int_0^1 f'(uX^\square) du \\ &= \sigma^2 \mathbb{E} \left[\frac{f(X^\square)}{X^\square} \right] \\ &= \mathbb{E} \left[X^2 \frac{f(X)}{X} \right] \\ &= \mathbb{E}[X f(X)] \end{aligned}$$

where X^\square and U are independent, $U \sim \mathbf{U}[0, 1]$ and X^\square has the X -square bias distribution. Thus, using (5), we have

$$\sigma^2 \mathbb{E}g(UX^\square) = \mathbb{E}[X f(X)] = \sigma^2 \mathbb{E}[f'(X^*)] = \sigma^2 \mathbb{E}[g(X^*)].$$

Since the expectation of $g(X^*)$ and $g(UX^\square)$ agree for any $g \in C_c$, with $=_d$ denoting distributional equivalence, we obtain

$$X^* =_d UX^\square.$$

Smoothing X by adding an independent random variable Y having the X -zero bias distribution, we obtain the following result which will be used for constructing Stein coefficients for sums.

Lemma 2.4. *If X is a mean zero random variable with finite non-zero variance, and Y is an independent variable with the X -zero bias distribution, then*

$$\mathbb{E}[X f(X + Y)] = \mathbb{E}[(X^2 - XY) f'(X + Y)]$$

for all Lipschitz functions f and a.e. derivative f' for which these expectations exist.

Proof. Let V be distributed as X , let U be a $\mathbf{U}[0, 1]$ random variable, and set

$$Y = UV^\square$$

where V, U, V^\square and X are independent. Note that for any bivariate function g for which the expectations below exist, by (4) we have

$$\mathbb{E}[g(X, V^\square)] = \frac{1}{\sigma^2} \mathbb{E}[V^2 g(X, V)], \quad (6)$$

where σ^2 is the variance of X . Hence

$$\begin{aligned} & \mathbb{E}[(X^2 - XY)f'(X + Y)] \\ &= \mathbb{E}[(X^2 - XUV^\square)f'(X + UV^\square)] \\ &= \mathbb{E}\left[\int_0^1 (X^2 - XuV^\square)f'(X + uV^\square)du\right] \\ &= \mathbb{E}\left[\left.\frac{(X^2 - XuV^\square)f(X + uV^\square)}{V^\square}\right|_0^1 + XV^\square \int_0^1 \frac{f(X + uV^\square)}{V^\square} du\right] \\ &= \mathbb{E}\left[\frac{(X^2 - XV^\square)f(X + V^\square) - X^2f(X)}{V^\square}\right] + \mathbb{E}[Xf(X + Y)] \\ &= \frac{1}{\sigma^2} \mathbb{E}[V(X^2 - XV)f(X + V) - VX^2f(X)] + \mathbb{E}[Xf(X + Y)] \\ &= \frac{1}{\sigma^2} \mathbb{E}[VX(X - V)f(X + V)] + \mathbb{E}[Xf(X + Y)], \end{aligned}$$

where we have used (6) in the second to last equality, as well as the independence of V and X , and that $\mathbb{E}V = 0$, in the last. Hence, to prove the claim, it suffices to show that the first term above is zero. Since $X =_d V$ and V and X are independent and exchangeable, we have

$$VX(X - V)f(X + V) =_d VX(V - X)f(X + V) = -VX(X - V)f(X + V),$$

demonstrating that the expectation of the expression above is zero. \square

For any mean zero X with finite, non-zero variance σ^2 the distribution of X^* is absolutely continuous with density function

$$p_{X^*}(x) = \frac{\mathbb{E}[X\mathbf{1}(X > x)]}{\sigma^2}. \quad (7)$$

One finds directly from (7) that

$$a \leq X \leq b \text{ for some constants } a < b \text{ implies } a \leq X^* \leq b. \quad (8)$$

Comparing (3) with (5), we see that T is a Stein coefficient for X if $\sigma^{-2}E[T|X]$ is the Radon Nikodym derivative $\frac{d\mu^*}{d\mu}$ of the probability measure μ^* of X^* with respect to the measure μ of X . Hence, in light of (7), if X is a random variable with mean zero and finite variance, having density function $p_X(x)$ whose support is an interval, then setting

$$h_X(x) = \frac{\mathbb{E}[X\mathbf{1}(X > x)]}{p_X(x)} \mathbf{1}(p_X(x) > 0) \quad \text{we have} \quad \mathbb{E}[Xf(X)] = \mathbb{E}[h_X(X)f'(X)] \quad (9)$$

for all Lipschitz function f and a.e. derivative f' for which these expectations exist, that is, $h_X(X)$ is a Stein coefficient for X . We note the first equality in (9) shows, by virtue of $\mathbb{E}(X) = 0$, that $h_X(x) \geq 0$.

Now consider a random variable X having vanishing first and third moment, variance strictly between zero and infinity and satisfying $\mathbb{E}(X^4) < \infty$. Then the distribution for a random variable Y having the X -zero bias distribution exists, and from (5) with $g(x) = x^2$ and $g(x) = x^3$, we respectively find

$$\mathbb{E}(Y) = 0 \quad \text{and} \quad \mathbb{E}(Y^2) < \infty. \quad (10)$$

Moreover from (7) we see that Y has density function $p_Y(y)$ whose support is a closed interval. Hence the function $h_Y(y)$, given by the first equality in (9), satisfies the second.

Lemma 2.5. *Let $\varepsilon_1, \dots, \varepsilon_n$ be independent and identically distributed as ε , a random variable with mean zero, finite nonzero variance, and satisfying $\mathbb{E}(\varepsilon^3) = 0$ and $\mathbb{E}(\varepsilon^4) < \infty$, and let Y have the ε -zero bias distribution and be independent of $\varepsilon_1, \dots, \varepsilon_n$. Then for all Lipschitz functions f and a.e. derivative f' ,*

$$\mathbb{E}[\tilde{S}_n f(\tilde{S}_n)] = \mathbb{E}[T f'(\tilde{S}_n)]$$

where

$$\tilde{S}_n = S_n + Y \quad \text{with} \quad S_n = \varepsilon_1 + \varepsilon_2 + \dots + \varepsilon_n,$$

and

$$T = \sum_{i=1}^n \varepsilon_i^2 - S_n Y + h_Y(Y) \quad \text{with} \quad h_Y(y) = \frac{\mathbb{E}[Y \mathbf{1}(Y > y)]}{p_Y(y)} \mathbf{1}(p_Y(y) > 0).$$

Proof. With $S_n^{(i)} = S_n - \varepsilon_i$, we have

$$\mathbb{E}[\tilde{S}_n f(\tilde{S}_n)] = \mathbb{E}[S_n f(\tilde{S}_n) + Y f(\tilde{S}_n)] = \sum_{i=1}^n \mathbb{E}[\varepsilon_i f(\varepsilon_i + Y + S_n^{(i)})] + \mathbb{E}[Y f(Y + S_n)]. \quad (11)$$

For the first term of (11), using that the summands ε_i are independent and applying Lemma 2.4 yields

$$\mathbb{E}[\varepsilon_i f(\varepsilon_i + Y + S_n^{(i)})] = \mathbb{E}[(\varepsilon_i^2 - \varepsilon_i Y) f'(\varepsilon_i + Y + S_n^{(i)})] = \mathbb{E}[(\varepsilon_i^2 - \varepsilon_i Y) f'(\tilde{S}_n)].$$

Now turning to the second term of (11), we first note that by (5) the assumption that the third moment of ε is zero implies $E(Y) = 0$. Now using the independence of Y and S_n , (9) yields

$$\mathbb{E}[Y f(Y + S_n)] = \mathbb{E}[h_Y(Y) f'(Y + S_n)] = \mathbb{E}[h_Y(Y) f'(\tilde{S}_n)].$$

Substitution into (11) now yields the claim. \square

Hoeffding's lemma, e.g. see the proof of Lemma 2.2 of [4], will be used below. It states that if X is a mean zero random variable that satisfies $a \leq X \leq b$ almost surely, then

$$\mathbb{E}[\exp(\theta X)] \leq e^{(b-a)^2 \theta^2 / 8} \quad \text{for all } \theta \in \mathbb{R}. \quad (12)$$

We also require the ‘non central χ_1^2 ’ moment generating function identity,

$$\mathbb{E} \exp(\alpha V^2 + \beta V) = \frac{\exp\left(\frac{\beta^2}{2(1-2\alpha)}\right)}{(1-2\alpha)^{1/2}} \quad (13)$$

valid for the standard Gaussian variable V , and all $\beta \in \mathbb{R}$ and $\alpha < 1/2$.

For the law $\mathcal{L}(X)$ of any random variable X let

$$\ell(\mathcal{L}(X)) = \inf\{b - a : P(a \leq X \leq b) = 1\},$$

the length of the support of X . For notational simplicity we will write $\ell(X)$, or ℓ when X is clear from context, for $\ell(\mathcal{L}(X))$. We use that $\ell(X)$ is translation invariant in the sense that $\ell(X) = \ell(X - c)$ for any real number c without further mention.

Lemma 2.6. *For every almost surely bounded random variable X , there exists a constant $\vartheta_{\ell(X)} \in (0, \infty)$ depending only on $\ell(X)$ such that when X_1, X_2, \dots are independent random variables distributed as X , the sum $S_n = X_1 + \dots + X_n$ and $\mu = \mathbb{E}X$ satisfy*

$$\mathbb{E} \left[\exp \left(\theta^2 \frac{S_n^2}{n} \right) \right] \leq \frac{4}{3} \exp \left(\frac{4}{3} n \theta^2 \mu^2 \right) \quad \text{for all } n \geq 1 \text{ and } |\theta| \leq \vartheta_{\ell(X)}.$$

The constant $4/3$ is somewhat arbitrary as any value greater than 1 can be achieved; the proof of Theorem 3.1 requires a value strictly less than $3/2$.

Proof. Let V be a $\mathcal{N}(0, 1)$ random variable independent of X . Using Hoeffding’s lemma (12) conditional on V , for any function of V we have

$$\mathbb{E}[\exp(t(V)(X - \mu)) | V] \leq e^{\ell^2 t(V)^2 / 8}.$$

Applying $\mathbb{E}(\exp \theta V) = \exp(\theta^2/2)$, for $\ell\theta < \sqrt{2}$ and V independent of X_1, X_2, \dots , letting $t(V) = \sqrt{2}\theta \frac{V}{\sqrt{n}}$ we obtain

$$\begin{aligned} \mathbb{E} \left[\exp \left(\theta^2 \frac{S_n^2}{n} \right) \right] &= \mathbb{E} \left[\exp \left(\sqrt{2}\theta \frac{S_n}{\sqrt{n}} V \right) \right] = \mathbb{E} \left[\mathbb{E} \left(\exp \left(\sqrt{2}\theta \frac{V}{\sqrt{n}} X \right) \middle| V \right)^n \right] \\ &= \mathbb{E} \left[\mathbb{E} \left(\exp(t(V)X) \middle| V \right)^n \right] = \mathbb{E} \left[\mathbb{E} \left(\exp(t(V)(X - \mu) + t(V)\mu) \middle| V \right)^n \right] \\ &\leq \mathbb{E} \left[\exp \left(\frac{2\ell^2 \theta^2 V^2}{8n} + \sqrt{2}\theta \mu \frac{V}{\sqrt{n}} \right)^n \right] = \mathbb{E} \left[\exp \left(\frac{\ell^2 \theta^2}{4} V^2 + \sqrt{2}\theta \mu \sqrt{n} V \right) \right] \\ &= \frac{1}{\sqrt{1 - \ell^2 \theta^2 / 2}} \exp \left(\frac{n \theta^2 \mu^2}{1 - \ell^2 \theta^2 / 2} \right) \leq \frac{1}{1 - \ell^2 \theta^2 / 2} \exp \left(\frac{n \theta^2 \mu^2}{1 - \ell^2 \theta^2 / 2} \right), \end{aligned}$$

where we have applied (13) in the last line. It is now direct to verify that the property required by the lemma holds by letting $\vartheta_{\ell(X)} = 1/(\sqrt{2}\ell(X))$, the unique positive solution to

$$\frac{1}{1 - \ell(X)^2 \theta^2 / 2} = \frac{4}{3}.$$

□

Lemma 2.7. *Let ε be a bounded, mean zero, variance $\sigma^2 \in (0, \infty)$ random variable satisfying $\mathbb{E}\varepsilon^3 = 0$. Then the Stein coefficient $h_Y(y)$, given by (9) for Y with the ε -zero bias distribution, is bounded.*

Proof. As ε is a mean zero random variable with finite, nonzero variance, the zero bias distribution $\mathcal{L}(Y)$ exists. As $\mathbb{E}\varepsilon^3 = 0$ and ε is bounded and non-trivial, as in (10) one verifies that $\mathbb{E}Y = 0$ and that $\text{Var}(Y)$ is positive and finite. Hence, as noted below (9), the Stein coefficient $h_Y(y)$ as given by (9) is nonnegative, so we need only show that it is bounded above.

From (7), an a.e. density of Y is given by

$$p_Y(y) = \frac{1}{\sigma^2} \int_y^\infty u dF_\varepsilon(u) \quad (14)$$

where we use F_X to denote the distribution function of the random variable X . From (14) we may observe that the support of Y is the smallest closed interval of \mathbb{R} containing the support of ε . Since ε is bounded and has mean zero, using (8), this interval is of the form $[a, b]$ for $-\infty < a < 0 < b < \infty$, hence for $t \in [a, b]$ the upper limit of the integral in (14) may be replaced by b .

In particular, for all $t \in [0, b]$, by (9) we have

$$\begin{aligned} h_Y(t) &= \frac{\int_t^b y p_Y(y) dy}{p_Y(t)} \\ &= \frac{\int_t^b y \int_y^b u dF_\varepsilon(u) dy}{\sigma^2 p_Y(t)} \\ &= \frac{\int \int_{t \leq y \leq u \leq b} y u dF_\varepsilon(u) dy}{\sigma^2 p_Y(t)} \\ &= \frac{\int_t^b u \int_t^u y dy dF_\varepsilon(u)}{\sigma^2 p_Y(t)} \\ &= \frac{\int_t^b u(u^2 - t^2) dF_\varepsilon(u)}{2\sigma^2 p_Y(t)} \\ &\leq \frac{b^2 \int_t^b u dF_\varepsilon(u)}{2\sigma^2 p_Y(t)} = \frac{b^2}{2}, \end{aligned}$$

where we have used Fubini's theorem in the fourth equality, and (14) in the second and sixth. As $h_{-Y}(t) = h_Y(-t)$ we obtain that $h_Y(y)$ is bounded for $t \in [a, 0]$. \square

Proof of Theorem 2.1: For short we write $S = \varepsilon_1 + \varepsilon_2 + \dots + \varepsilon_n$ and $\tilde{S} = S + Y$ with Y is as in Lemma 2.5. As the third moment of ε is zero and its fourth moment is finite, as in (10), Y has mean zero with finite variance, and hence so does \tilde{S} .

By Lemma 2.5, $T = \sum_{i=1}^n \varepsilon_i^2 - SY + h_Y(Y)$ is a Stein coefficient for \tilde{S} . Since ε is bounded and the third moment of ε is zero, Lemma 2.7 yields that $h_Y(Y)$ is bounded. Also ε bounded implies S is bounded. In addition, as ε is bounded there exists some B such that $|\varepsilon| \leq B$, and (8) implies $|Y| \leq B$. Thus, we conclude that $|T|$ is bounded.

Now invoking Theorem 2.3, there exists a version of \tilde{S} and $Z \sim \mathcal{N}(0, \sigma^2)$ on the same probability space such that

$$\mathbb{E} \exp(\theta |\tilde{S} - Z|) \leq 2\mathbb{E} (\exp(2\theta^2 \sigma^{-2} (T - \sigma^2)^2)) \quad \text{for all } \theta \in \mathbb{R}.$$

Using $|Y| \leq B$ we have $|S - \tilde{S}| \leq B$. It follows that,

$$\mathbb{E} \exp(\theta|S - Z|) \leq 2\mathbb{E} \left(\exp(B|\theta| + 2\theta^2 \sigma^{-2}(T - \sigma^2)^2) \right).$$

Letting $C_0 \geq B$ be such that $|h_Y(Y)| \leq C_0$, and setting $\sigma^2 = n$, we obtain

$$\frac{(T - \sigma^2)^2}{\sigma^2} \leq \frac{3\bar{S}^2 + 3C_0^2 S^2 + 3C_0^2}{n}$$

where $\bar{S} = \sum_{i=1}^n (\varepsilon_i^2 - 1)$.

Hence,

$$\begin{aligned} \mathbb{E} \exp(\theta|S - Z|) &\leq 2 \exp\left(B|\theta| + \frac{6C_0^2 \theta^2}{n}\right) \mathbb{E} \exp\left(6\theta^2 \frac{\bar{S}^2 + C_0^2 S^2}{n}\right) \\ &\leq \exp\left(B|\theta| + \frac{6C_0^2 \theta^2}{n}\right) \mathbb{E} \left[\exp\left(\frac{12\theta^2 \bar{S}^2}{n}\right) + \exp\left(\frac{12\theta^2 C_0^2 S^2}{n}\right) \right] \end{aligned} \quad (15)$$

where we applied the simple inequality $\exp(x + y) \leq (e^{2x} + e^{2y})/2$.

Noting for \bar{S} and S that $\varepsilon^2 - 1$ and ε respectively are bounded and have mean zero, using Lemma 2.6 for the first two inequalities below, we see that there exists $\theta_1 > 0$ such that for all $|\theta| \leq \theta_1$ and all positive integers n

$$\mathbb{E} \exp(12\theta^2 \bar{S}^2/n) \leq 2 \quad \text{and} \quad \mathbb{E} \exp(12\theta^2 C_0^2 S^2/n) \leq 2 \quad \text{and} \quad \exp\left(B|\theta| + \frac{6C_0^2 \theta^2}{n}\right) \leq 2.$$

Theorem 2.1 now follows from (15). \square

We now prepare for the proof of Theorem 2.2 by providing a few lemmas. For \mathcal{A} the finite set in which the basic variable ε takes values, let

$$\mathcal{D} = \{b - a : a, b \in \mathcal{A}\} \quad \text{and} \quad \mathcal{D}^+ = \mathcal{D} \cap [0, \infty), \quad (16)$$

the set of differences of the elements in \mathcal{A} , and those differences that are non-negative. We note here that \mathcal{D} is symmetric in that $\mathcal{D} = -\mathcal{D}$. Let also

$$B = \max_{a \in \mathcal{A}} |a|. \quad (17)$$

Recall the definition (2) of W_k and observe that we may write

$$\begin{aligned} W_k &= S_k - \frac{k}{n} S_n = \sum_{i=1}^k \varepsilon_{\pi(i)} - \frac{k}{n} \sum_{i=1}^n \varepsilon_{\pi(i)} = \frac{n-k}{n} \sum_{i=1}^k \varepsilon_{\pi(i)} - \frac{k}{n} \sum_{i=k+1}^n \varepsilon_{\pi(i)} \\ &= \frac{1}{n} \sum_{i=1}^k \sum_{j=k+1}^n (\varepsilon_{\pi(i)} - \varepsilon_{\pi(j)}), \end{aligned} \quad (18)$$

and therefore

$$W_k = \sum_{d \in \mathcal{D}^+} W_{k,d} \quad \text{where} \quad W_{k,d} = \frac{1}{n} \sum_{i=1}^k \sum_{j=k+1}^n (\varepsilon_{\pi(i)} - \varepsilon_{\pi(j)}) \mathbb{1}_{(|\varepsilon_{\pi(i)} - \varepsilon_{\pi(j)}| = d)}. \quad (19)$$

Lemma 2.8. *Under the hypotheses of Theorem 2.2, for any $\theta \in \mathbb{R}$, $1 \leq k \leq n$ and $d \in \mathcal{D}^+$ we have*

$$\mathbb{E} \exp(\theta W_{k,d}/\sqrt{k}) \leq \exp(d^2 \theta^2/2) \quad \text{and} \quad \mathbb{E} \exp(\theta W_k/\sqrt{k}) \leq \exp(B^2 \theta^2), \quad (20)$$

where B is as in (17). Further, there exists $\alpha_0 > 0$ depending only on \mathcal{A} such that

$$\mathbb{E}[\exp(\alpha W_{k,d}^2/k)] \leq 2 \quad \text{for all } |\alpha| \leq \alpha_0 \text{ and all } d \in \mathcal{D}^+. \quad (21)$$

Proof. We may assume $d > 0$ as the result is otherwise trivial. Fix an integer k in $[1, n]$ and $d \in \mathcal{D}^+$, and let $m(\theta) := \mathbb{E} \exp(\theta W_{k,d}/\sqrt{k})$. We argue as in [6]. Since $W_{k,d}$ is bounded, the function $m(\theta)$ is differentiable and differentiation and expectation may be interchanged. Hence, using (19) for the second equality,

$$\begin{aligned} m'(\theta) &= \frac{1}{\sqrt{k}} \mathbb{E}(W_{k,d} \exp(\theta W_{k,d}/\sqrt{k})) \\ &= \frac{1}{n\sqrt{k}} \sum_{i=1}^k \sum_{j=k+1}^n \mathbb{E}[(\varepsilon_{\pi(i)} - \varepsilon_{\pi(j)}) \mathbb{1}_{(|\varepsilon_{\pi(i)} - \varepsilon_{\pi(j)}|=d)} \exp(\theta W_{k,d}/\sqrt{k})]. \end{aligned} \quad (22)$$

Now, let i and j satisfying $1 \leq i \leq k < j \leq n$ be arbitrary and let $\pi' = \pi \circ (i, j)$ where (i, j) is the transposition of i and j . Then (π, π') is an exchangeable pair of random permutations. Let $W'_{k,d}$ be defined as in (19) with π' replacing π . Using exchangeability for the first equality and the definition of π' for the second,

$$\begin{aligned} &\mathbb{E}[(\varepsilon_{\pi(i)} - \varepsilon_{\pi(j)}) \mathbb{1}_{(|\varepsilon_{\pi(i)} - \varepsilon_{\pi(j)}|=d)} \exp(\theta W_{k,d}/\sqrt{k})] \\ &= \mathbb{E}[(\varepsilon_{\pi'(i)} - \varepsilon_{\pi'(j)}) \mathbb{1}_{(|\varepsilon_{\pi'(i)} - \varepsilon_{\pi'(j)}|=d)} \exp(\theta W'_{k,d}/\sqrt{k})] \\ &= \mathbb{E}[(\varepsilon_{\pi(j)} - \varepsilon_{\pi(i)}) \mathbb{1}_{(|\varepsilon_{\pi(i)} - \varepsilon_{\pi(j)}|=d)} \exp(\theta W'_{k,d}/\sqrt{k})] \\ &= - \mathbb{E}[(\varepsilon_{\pi(i)} - \varepsilon_{\pi(j)}) \mathbb{1}_{(|\varepsilon_{\pi(i)} - \varepsilon_{\pi(j)}|=d)} \exp(\theta W'_{k,d}/\sqrt{k})]. \end{aligned}$$

Averaging the first and last expressions yields

$$\begin{aligned} &\mathbb{E}[(\varepsilon_{\pi(i)} - \varepsilon_{\pi(j)}) \mathbb{1}_{(|\varepsilon_{\pi(i)} - \varepsilon_{\pi(j)}|=d)} \exp(\theta W_{k,d}/\sqrt{k})] \\ &= \frac{1}{2} \mathbb{E}[(\varepsilon_{\pi(i)} - \varepsilon_{\pi(j)}) \mathbb{1}_{(|\varepsilon_{\pi(i)} - \varepsilon_{\pi(j)}|=d)} (\exp(\theta W_{k,d}/\sqrt{k}) - \exp(\theta W'_{k,d}/\sqrt{k}))]. \end{aligned} \quad (23)$$

Note

$$\begin{aligned} &|W_{k,d} - W'_{k,d}| \\ &= \frac{1}{n} \left| \sum_{k+1 \leq l \leq n, l \neq j} (\varepsilon_{\pi(i)} - \varepsilon_{\pi(l)}) \mathbb{1}_{(|\varepsilon_{\pi(i)} - \varepsilon_{\pi(l)}|=d)} + \sum_{1 \leq l \leq k, l \neq i} (\varepsilon_{\pi(l)} - \varepsilon_{\pi(j)}) \mathbb{1}_{(|\varepsilon_{\pi(l)} - \varepsilon_{\pi(j)}|=d)} \right. \\ &\quad \left. + (\varepsilon_{\pi(i)} - \varepsilon_{\pi(j)}) \mathbb{1}_{(|\varepsilon_{\pi(i)} - \varepsilon_{\pi(j)}|=d)} - \left(\sum_{k+1 \leq l \leq n, l \neq j} (\varepsilon_{\pi'(i)} - \varepsilon_{\pi'(l)}) \mathbb{1}_{(|\varepsilon_{\pi'(i)} - \varepsilon_{\pi'(l)}|=d)} \right. \right. \\ &\quad \left. \left. + \sum_{1 \leq l \leq k, l \neq i} (\varepsilon_{\pi'(l)} - \varepsilon_{\pi'(j)}) \mathbb{1}_{(|\varepsilon_{\pi'(l)} - \varepsilon_{\pi'(j)}|=d)} + (\varepsilon_{\pi'(i)} - \varepsilon_{\pi'(j)}) \mathbb{1}_{(|\varepsilon_{\pi'(i)} - \varepsilon_{\pi'(j)}|=d)} \right) \right| \\ &= \frac{1}{n} \left| \sum_{l=1}^n (\varepsilon_{\pi(i)} - \varepsilon_{\pi(l)}) \mathbb{1}_{(|\varepsilon_{\pi(i)} - \varepsilon_{\pi(l)}|=d)} + \sum_{l=1}^n (\varepsilon_{\pi(l)} - \varepsilon_{\pi(j)}) \mathbb{1}_{(|\varepsilon_{\pi(l)} - \varepsilon_{\pi(j)}|=d)} \right| \\ &\leq \frac{1}{n} [nd + nd] = 2d. \end{aligned}$$

Now applying the inequality $|e^x - e^y| \leq \frac{1}{2}|x - y|(e^x + e^y)$ we see that (23) in absolute value is bounded by

$$\begin{aligned} & \frac{|\theta|}{4\sqrt{k}} \mathbb{E}[|\varepsilon_{\pi(i)} - \varepsilon_{\pi(j)}| \mathbb{1}_{(|\varepsilon_{\pi(i)} - \varepsilon_{\pi(j)}|=d)} |W_{k,d} - W'_{k,d}| ((\exp(\theta W_{k,d}/\sqrt{k}) + \exp(\theta W'_{k,d}/\sqrt{k})))] \\ & \leq \frac{|\theta|}{4\sqrt{k}} 2d^2 \mathbb{E}[\exp(\theta W_{k,d}/\sqrt{k}) + \exp(\theta W'_{k,d}/\sqrt{k})] \\ & = \frac{|\theta|d^2}{\sqrt{k}} m(\theta). \end{aligned}$$

So, from (22), and the fact that $1 \leq i \leq k$ and $k < j \leq n$ are arbitrary, we obtain

$$|m'(\theta)| \leq \frac{1}{n\sqrt{k}} \frac{|\theta|d^2}{\sqrt{k}} \sum_{i=1}^k \sum_{j=k+1}^n m(\theta) \leq d^2 |\theta| m(\theta).$$

Now, using, $m(0) = 1$, and that $m(\theta) \geq 0$ for all $\theta \in \mathbb{R}$, for $\theta > 0$, we obtain

$$\int_0^\theta \frac{m'(u)}{m(u)} du \leq \int_0^\theta d^2 u du \implies m(\theta) \leq \exp(d^2 \theta^2 / 2)$$

and for $\theta < 0$, we obtain

$$\int_\theta^0 \frac{-m'(u)}{m(u)} du \leq \int_\theta^0 d^2 (-u) du \implies m(\theta) \leq \exp(d^2 \theta^2 / 2),$$

proving the first inequality of (20).

Arguing similarly, now letting $m(\theta) := \mathbb{E} \exp(\theta W_k / \sqrt{k})$ and W'_k as in (2) with π' replacing π , noting $|W_k - W'_k| = |\varepsilon_{\pi(i)} - e_{\pi(j)}| \leq 2B$, we obtain

$$\begin{aligned} \mathbb{E}[(\varepsilon_{\pi(i)} - \varepsilon_{\pi(j)}) \exp(\theta W_k / \sqrt{k})] & \leq \frac{|\theta|}{4\sqrt{k}} \mathbb{E}[(\varepsilon_{\pi(i)} - \varepsilon_{\pi(j)})^2 ((\exp(\theta W_k / \sqrt{k}) + \exp(\theta W'_k / \sqrt{k})))] \\ & \leq \frac{|\theta| 4B^2}{4\sqrt{k}} 2m(\theta) = \frac{2|\theta| B^2}{\sqrt{k}} m(\theta), \end{aligned}$$

so that $|m'(\theta)| \leq 2B^2 |\theta| m(\theta)$, implying the final inequality of (20).

Turning to (21), letting Z be a standard normal random variable independent of $W_{k,d}$, by (20) and (13), for all $d \in \mathcal{D}^+$ and $\alpha < 1/(2d^2)$, we have

$$\mathbb{E} \exp(\alpha W_{k,d}^2 / k) = \mathbb{E} \exp\left(\sqrt{2\alpha} Z W_{k,d} / \sqrt{k}\right) \leq \mathbb{E} \exp(d^2 \alpha Z^2) \leq \frac{1}{\sqrt{1-2d^2\alpha}}.$$

Now set α_0 so that the bound above is any number no greater than 2 when d is replaced by $\max\{d : d \in \mathcal{D}^+\}$. \square

Lemma 2.9. *Under the assumptions of Theorem 2.2 there exists $\alpha_1 > 0$ depending only on \mathcal{A} such that for all n , all $1 \leq k \leq 2n/3$, and all $0 \leq \alpha \leq \alpha_1$,*

$$\mathbb{E} \exp(\alpha S_k^2 / k) \leq \exp\left(1 + \frac{3\alpha S_n^2}{4n}\right).$$

Proof. The steps are the same as in the proof of Lemma 3.5 of [6]. For Z a standard normal random variable independent of π , by definition (2) of W_k we have

$$\begin{aligned}\mathbb{E} \exp(\alpha S_k^2/k) &= \mathbb{E} \exp \left(\sqrt{\frac{2\alpha}{k}} S_k Z \right) \\ &= \mathbb{E} \exp \left(\sqrt{\frac{2\alpha}{k}} W_k Z + \sqrt{\frac{2\alpha}{k}} \frac{k S_n}{n} Z \right).\end{aligned}$$

By (20), with B given by (17), for the first term we obtain the bound

$$\mathbb{E} \left[\exp \left(\sqrt{2\alpha} Z W_k / \sqrt{k} \right) \middle| Z \right] \leq \exp(2\alpha B^2 Z^2).$$

Thus,

$$\mathbb{E} \exp(\alpha S_k^2/k) \leq \mathbb{E} \exp \left(2\alpha B^2 Z^2 + \sqrt{\frac{2\alpha}{k}} \frac{k S_n}{n} Z \right).$$

Recalling S_n is nonrandom, using the non central χ_1^2 identity (13), we find that

$$\mathbb{E} \exp(\alpha S_k^2/k) \leq \frac{1}{\sqrt{1-4\alpha B^2}} \exp \left(\frac{\alpha k S_n^2}{(1-4\alpha B^2)n^2} \right) \quad \text{for } 0 < \alpha < 1/(4B^2).$$

The proof of the lemma is now completed by bounding k by $2n/3$ and choosing $\alpha_1 > 0$ small enough so that $1/(1-4\alpha_1 B^2)$ is sufficiently close to 1. \square

Proof of Theorem 2.2: We assume $\theta > 0$. Applying our convention that zero variance normal random variables are equal to their mean almost surely, when $n = 1$ we have $S_0 = W_0 = W_1 = Z_0 = Z_1 = 0$ and the result holds trivially, so we assume $n \geq 2$. Recalling definition (16) of \mathcal{D}^+ for each $d > 0$ in \mathcal{D}^+ and that \mathcal{D} is symmetric, let Y_d have the uniform $\mathbf{U}[-d/2, d/2]$ distribution, and be independent of each other and of the uniform random permutation π , and for $d = 0$ let $Y_0 = 0$. Set

$$Y = \sum_{d \in \mathcal{D}^+} Y_d.$$

For arbitrary i, j satisfying $1 \leq i \leq k < j \leq n$ let $\mathcal{F}_{ij} = \sigma\{\pi(l) : l \notin \{i, j\}\}$. Regarding the collection $\{\varepsilon_1, \dots, \varepsilon_n\}$ as a multiset, we have

$$\{\varepsilon_{\pi(i)}, \varepsilon_{\pi(j)}\} = \{\varepsilon_i, i = 1, \dots, n\} \setminus \{\varepsilon_{\pi(l)}, l \notin \{i, j\}\},$$

showing that $\{\varepsilon_{\pi(i)}, \varepsilon_{\pi(j)}\}$, and therefore also $\varepsilon_{\pi(i)} + \varepsilon_{\pi(j)}$ and $d_{ij} := |\varepsilon_{\pi(i)} - \varepsilon_{\pi(j)}|$ are measurable with respect to \mathcal{F}_{ij} . Further, the conditional distribution of

$$X_{ij} := \frac{\varepsilon_{\pi(i)} - \varepsilon_{\pi(j)}}{2}$$

given \mathcal{F}_{ij} is uniform over the set $\{-d_{ij}/2, d_{ij}/2\}$.

Let $S_k^{(i)} = S_k - \varepsilon_{\pi(i)}$, $W_k^{(i)} = S_k^{(i)} - (k/n)S_n$ and $Y^{(ij)} = Y - Y_{d_{ij}}$. For $\varepsilon_{\pi(i)} \neq \varepsilon_{\pi(j)}$, applying Lemma 2.4 and the easily verified fact that the zero bias distribution of the variable that takes the values $\{-a, a\}$ with equal probability is uniform over $[-a, a]$, for some fixed Lipschitz function f , we have

$$\begin{aligned}
& \mathbb{E}[(\varepsilon_{\pi(i)} - \varepsilon_{\pi(j)})f(W_k + Y)|\mathcal{F}_{ij}] \\
&= 2\mathbb{E}[X_{ij}f(X_{ij} + Y_{d_{ij}} + W_k^{(i)} + (\varepsilon_{\pi(i)} + \varepsilon_{\pi(j)})/2 + Y^{(ij)})|\mathcal{F}_{ij}] \\
&= 2\mathbb{E}[(X_{ij}^2 - X_{ij}Y_{d_{ij}})f'(X_{ij} + Y_{d_{ij}} + W_k^{(i)} + (\varepsilon_{\pi(i)} + \varepsilon_{\pi(j)})/2 + Y^{(ij)})|\mathcal{F}_{ij}] \\
&= 2\mathbb{E}[(X_{ij}^2 - X_{ij}Y_{d_{ij}})f'(W_k + Y)|\mathcal{F}_{ij}] \\
&= 2\mathbb{E}[(d_{ij}^2/4 - X_{ij}Y_{d_{ij}})f'(W_k + Y)|\mathcal{F}_{ij}].
\end{aligned}$$

We note that the equality between the first and final terms above holds also when $\varepsilon_{\pi(i)} = \varepsilon_{\pi(j)}$, both sides being zero. Taking expectation we obtain

$$\mathbb{E}[(\varepsilon_{\pi(i)} - \varepsilon_{\pi(j)})f(W_k + Y)] = \mathbb{E}[t_{ij}f'(W_k + Y)] \quad (24)$$

where

$$t_{ij} = 2 \left(\frac{d_{ij}^2}{4} - X_{ij}Y_{d_{ij}} \right) = \frac{\varepsilon_{\pi(i)}^2 + \varepsilon_{\pi(j)}^2}{2} - \varepsilon_{\pi(i)}\varepsilon_{\pi(j)} - (\varepsilon_{\pi(i)} - \varepsilon_{\pi(j)})Y_{d_{ij}}. \quad (25)$$

It is easy to verify using (9), or by integration by parts, that for $U \sim \mathbf{U}[-a, a]$,

$$\mathbb{E}[Uf(U)] = \frac{1}{2}\mathbb{E}[(a^2 - U^2)f'(U)],$$

implying

$$\begin{aligned}
\mathbb{E}[Yf(W_k + Y)] &= \sum_{d \in \mathcal{D}^+} \mathbb{E}[Y_d f(Y_d + W_k + (Y - Y_d))] \\
&= \frac{1}{2} \sum_{d \in \mathcal{D}^+} \mathbb{E} \left[\left(\frac{d^2}{4} - Y_d^2 \right) f'(Y_d + W_k + (Y - Y_d)) \right] = \mathbb{E}[R_4 f'(W_k + Y)], \quad (26)
\end{aligned}$$

where

$$R_4 = \frac{1}{2} \sum_{d \in \mathcal{D}^+} \left(\frac{d^2}{4} - Y_d^2 \right).$$

Since \mathcal{D}^+ is finite there exists $C_0 > 0$ so that

$$|R_4| \leq C_0. \quad (27)$$

From (18),

$$W_k = \frac{1}{n} \sum_{i=1}^k \sum_{j=k+1}^n (\varepsilon_{\pi(i)} - \varepsilon_{\pi(j)}),$$

so letting

$$\tilde{W}_k = W_k + Y \quad (28)$$

and combining (24) and (26), we have

$$\mathbb{E}[\tilde{W}_k f(\tilde{W}_k)] = \mathbb{E}[T f'(\tilde{W}_k)], \quad (29)$$

where the Stein coefficient T , in light of (25), is given by

$$T = \frac{1}{n} \sum_{i=1}^k \sum_{j=k+1}^n t_{ij} + R_4 = R_1 - R_2 - R_3 + R_4,$$

where

$$R_1 = \frac{1}{2n} \left((n-k) \sum_{i=1}^k \varepsilon_{\pi(i)}^2 + k \sum_{j=k+1}^n \varepsilon_{\pi(j)}^2 \right), \quad R_2 = \frac{1}{n} \sum_{i=1}^k \varepsilon_{\pi(i)} \sum_{j=k+1}^n \varepsilon_{\pi(j)},$$

and

$$\begin{aligned} R_3 &= \frac{1}{n} \sum_{1 \leq i \leq k < j \leq n} (\varepsilon_{\pi(i)} - \varepsilon_{\pi(j)}) Y_{d_{ij}} \\ &= \sum_{d \in \mathcal{D}^+} Y_d \frac{1}{n} \sum_{1 \leq i \leq k < j \leq n} (\varepsilon_{\pi(i)} - \varepsilon_{\pi(j)}) \mathbb{1}(|\varepsilon_{\pi(i)} - \varepsilon_{\pi(j)}| = d) = \sum_{d \in \mathcal{D}^+} Y_d W_{k,d}, \end{aligned}$$

with $W_{k,d}$ as in (19). Since $|Y_d| \leq d/2$, we have

$$|R_3| \leq \sum_{d \in \mathcal{D}^+} \frac{d}{2} |W_{k,d}|. \quad (30)$$

Recalling that $\nu > 0$ is a given fixed number, and that

$$\gamma^2 = \frac{1}{n} \sum_{i=1}^n \varepsilon_i^2, \quad \text{set} \quad \tilde{\sigma}^2 = \frac{k(n-k)\gamma^2}{n} \quad \text{and} \quad \sigma^2 = \frac{k(n-k)\eta^2}{n},$$

for a positive constant $\eta \geq \nu$, noting that since $n \geq 2$ both σ^2 and $\tilde{\sigma}^2$ are positive. Then,

$$\begin{aligned} \frac{(T - \sigma^2)^2}{\sigma^2} &= \frac{n}{k(n-k)\eta^2} (R_1 - \sigma^2 - R_2 - R_3 + R_4)^2 \\ &\leq \frac{n}{k(n-k)\nu^2} (R_1 - \sigma^2 - R_2 - R_3 + R_4)^2. \quad (31) \end{aligned}$$

To bound this quantity, consider first

$$\begin{aligned}
& R_1 - \tilde{\sigma}^2 \\
&= \frac{1}{2n} \left((n-k) \sum_{i=1}^k \varepsilon_{\pi(i)}^2 + k \sum_{j=k+1}^n \varepsilon_{\pi(j)}^2 \right) - \tilde{\sigma}^2 \\
&= \frac{1}{2n} \left((n-k) \sum_{i=1}^k \varepsilon_{\pi(i)}^2 + k \sum_{j=k+1}^n \varepsilon_{\pi(j)}^2 \right) - \frac{k(n-k)}{n^2} \sum_{i=1}^n \varepsilon_{\pi(i)}^2 \\
&= \frac{1}{2n} \left((n-k) \sum_{i=1}^k \varepsilon_{\pi(i)}^2 + k \sum_{j=k+1}^n \varepsilon_{\pi(j)}^2 - \frac{2k(n-k)}{n} \sum_{i=1}^n \varepsilon_{\pi(i)}^2 \right) \\
&= \frac{1}{2n} \left((n-k) \left(\sum_{i=1}^k \varepsilon_{\pi(i)}^2 - \frac{k}{n} \sum_{i=1}^n \varepsilon_{\pi(i)}^2 \right) + k \left(\sum_{j=k+1}^n \varepsilon_{\pi(j)}^2 - \frac{n-k}{n} \sum_{j=1}^n \varepsilon_{\pi(j)}^2 \right) \right) \\
&= \frac{1}{2n} \left((n-k) \left(\frac{n-k}{n} \sum_{i=1}^k \varepsilon_{\pi(i)}^2 - \frac{k}{n} \sum_{i=k+1}^n \varepsilon_{\pi(i)}^2 \right) + k \left(\frac{k}{n} \sum_{j=k+1}^n \varepsilon_{\pi(j)}^2 - \frac{n-k}{n} \sum_{j=1}^k \varepsilon_{\pi(j)}^2 \right) \right) \\
&= \frac{1}{2n^2} \left(((n-k)^2 - k(n-k)) \sum_{i=1}^k \varepsilon_{\pi(i)}^2 - ((n-k)k - k^2) \sum_{j=k+1}^n \varepsilon_{\pi(j)}^2 \right) \\
&= \frac{n-2k}{2n^2} \left((n-k) \sum_{i=1}^k \varepsilon_{\pi(i)}^2 - k \sum_{j=k+1}^n \varepsilon_{\pi(j)}^2 \right).
\end{aligned}$$

Hence, for all $k = 1, 2, \dots, n$, with B as in (17),

$$\begin{aligned}
|R_1 - \sigma^2| &\leq \frac{|n-2k|}{2n^2} \left((n-k) \sum_{i=1}^k \varepsilon_{\pi(i)}^2 + k \sum_{i=k+1}^n \varepsilon_{\pi(i)}^2 \right) + |\tilde{\sigma}^2 - \sigma^2| \\
&\leq \frac{|n-2k|}{2} \gamma^2 + |\tilde{\sigma}^2 - \sigma^2| \leq \frac{|n-2k|}{2} B^2 + |\tilde{\sigma}^2 - \sigma^2|.
\end{aligned}$$

Choosing k such that $|n-2k| \leq 1$, we obtain

$$|R_1 - \sigma^2| \leq \frac{B^2}{2} + |\tilde{\sigma}^2 - \sigma^2|. \quad (32)$$

Regarding R_2 , for any $k \in \{1, \dots, n\}$ we have

$$|R_2| = \frac{1}{n} \left| \sum_{i=1}^k \varepsilon_{\pi(i)} \sum_{j=k+1}^n \varepsilon_{\pi(j)} \right| \leq B |S_k|. \quad (33)$$

Hence, for k such that $|2k-n| \leq 1$, from (31), (32), (33), (30) and (27), and that $|\tilde{\sigma}^2 - \sigma^2| = \left| \frac{k(n-k)}{n} (\gamma^2 - \eta^2) \right|$, we obtain

$$\begin{aligned}
\frac{(T - \sigma^2)^2}{\sigma^2} &\leq \frac{n}{k(n-k)v^2} \left(B^2/2 + |\tilde{\sigma}^2 - \sigma^2| + B|S_k| + \sum_{d \in \mathcal{D}^+} \frac{d}{2} |W_{k,d}| + C_0 \right)^2 \\
&\leq C \left(1 + n(\gamma^2 - \eta^2)^2 + \frac{S_k^2}{k} + \sum_{d \in \mathcal{D}^+} \frac{W_{k,d}^2}{k} \right)
\end{aligned}$$

for some constant C depending uniquely on \mathcal{A} and \mathbf{v} .

We now verify that the hypotheses of Theorem 2.3 hold for \tilde{W}_k of (28) and T . Clearly \tilde{W}_k satisfies $\mathbb{E}(\tilde{W}_k) = 0$ and $\mathbb{E}(\tilde{W}_k^2) < \infty$. By (29), T is a Stein coefficient for \tilde{W}_k , and T is easily verified to be bounded. Writing Z for short for ηZ_k in the statement of Theorem 2.2, we note that Z is distributed $\mathcal{N}(0, \sigma^2)$, and by Theorem 2.3 we can construct a version of \tilde{W}_k and Z on the same probability space so that for all θ ,

$$\begin{aligned} \mathbb{E} \exp(\theta |\tilde{W}_k - Z|) &\leq 2 \mathbb{E} \exp(2\theta^2 \sigma^{-2} (T - \sigma^2)^2) \\ &\leq 2 \mathbb{E} \exp \left(2C\theta^2 \left(1 + n(\gamma^2 - \eta^2)^2 + \frac{S_k^2}{k} + \sum_{d \in \mathcal{D}^+} \frac{W_{k,d}^2}{k} \right) \right). \end{aligned}$$

With $D = \frac{1}{2} \sum_{d \in \mathcal{D}^+} d$ we have $|W_k - \tilde{W}_k| \leq |Y| \leq \sum_{d \in \mathcal{D}^+} |Y_d| \leq D$. Letting $q = |\mathcal{D}^+| + 1$, we have

$$\begin{aligned} \mathbb{E} \exp(\theta |W_k - Z|) &\leq 2 \exp(D|\theta| + 2C\theta^2 + 2C\theta^2 n(\gamma^2 - \eta^2)^2) \mathbb{E} \exp \left(2C\theta^2 \frac{S_k^2}{k} + 2C\theta^2 \sum_{d \in \mathcal{D}^+} \frac{W_{k,d}^2}{k} \right) \\ &\leq \frac{2}{q} \exp(D|\theta| + 2C\theta^2 + 2C\theta^2 n(\gamma^2 - \eta^2)^2) \left(\mathbb{E} \exp \left(2Cq\theta^2 \frac{S_k^2}{k} \right) + \sum_{d \in \mathcal{D}^+} \mathbb{E} \exp \left(2Cq\theta^2 \frac{W_{k,d}^2}{k} \right) \right), \end{aligned}$$

by the convexity of the exponential function. Using Lemmas 2.9 and 2.8, there exists $\theta_3 > 0$ depending only on \mathcal{A} and \mathbf{v} such that for all $\theta \leq \theta_3$, we obtain

$$\begin{aligned} \mathbb{E} \exp(\theta |W_k - Z|) &\leq \frac{2}{q} \exp(D|\theta| + 2C\theta^2 + 2C\theta^2 n(\gamma^2 - \eta^2)^2) \left(\exp \left(1 + 6Cq\theta^2 \frac{S_n^2}{4n} \right) + 2(q-1) \right) \\ &\leq 2 \exp(D|\theta| + 2C\theta^2 + 2C\theta^2 n(\gamma^2 - \eta^2)^2) \left(\exp \left(1 + 6Cq\theta^2 \frac{S_n^2}{4n} \right) + 2 \right). \end{aligned}$$

Now choose $\theta_4 > 0$, depending only on \mathcal{A} and \mathbf{v} , so that

$$2 \exp(D\theta_4 + 2C\theta_4^2) \leq e \quad \text{and note} \quad \exp(1 + \theta^2 x) + 2 \leq \exp(2 + \theta^2 x) \quad \text{for all } x \geq 0,$$

implying that for $\theta \leq \theta_2 := \theta_3 \wedge \theta_4$,

$$\begin{aligned} \mathbb{E} \exp(\theta |W_k - Z|) &\leq \exp \left(1 + 2C\theta^2 n(\gamma^2 - \eta^2)^2 + 2 + 6Cq\theta^2 \frac{S_n^2}{4n} \right) \\ &= \exp \left(3 + 6Cq\theta^2 \frac{S_n^2}{4n} + 2C\theta^2 n(\gamma^2 - \eta^2)^2 \right), \end{aligned}$$

which is the desired bound. \square

3 The Induction Step

In this section we present Theorem 3.1, which we use to prove Theorem 1.4 that generalizes Theorem 1.4 in [6]. Let $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n$ be arbitrary elements of a finite set $\mathcal{A} \subset \mathbb{R}$, not necessarily distinct, and let π be a uniform random permutation of $\{1, 2, \dots, n\}$. For each $1 \leq k \leq n$ recall

$$S_k = \sum_{i=1}^k \varepsilon_{\pi(i)} \quad \text{and} \quad W_k = S_k - \frac{kS_n}{n}. \quad (34)$$

We show (W_1, \dots, W_n) and a positive multiple of a Gaussian vector (Z_1, \dots, Z_n) obtained by evaluating a Brownian bridge process on $[0, n]$ at integer time points can be coupled on the same space so that the moment generating function of their maximum absolute difference achieves the exponential bound (39) below. In place of coupling, the result of the theorem can be equivalently stated in terms of the existence of a joint probability function $\rho_{\boldsymbol{\varepsilon}}^n(\mathbf{s}, \mathbf{z})$ on (S_1, \dots, S_n) and (Z_1, \dots, Z_n) having the correct marginals whose joint realization obeys the desired bound.

It will be helpful to regard the collection $\boldsymbol{\varepsilon} = \{\varepsilon_1, \dots, \varepsilon_n\}$ as a multiset. We say $\mathbf{s} \in \mathbb{R}^n$ is a ‘path’ corresponding to a multiset of ‘increments’ $\boldsymbol{\varepsilon}$ when there exists $\pi \in \mathcal{P}_n$, the set of permutations on $\{1, \dots, n\}$, such that \mathbf{s} can be achieved by summing the increments ε in the order given by π , that is, when \mathbf{s} is an element of the set of all feasible paths

$$\mathcal{A}_{\boldsymbol{\varepsilon}}^n := \{\mathbf{s} \in \mathbb{R}^n : s_k = \sum_{i=1}^k \varepsilon_{\pi(i)}, k = 1, \dots, n, \pi \in \mathcal{P}_n\}. \quad (35)$$

Conversely, the multiset of increments corresponding to a path \mathbf{s} is given by

$$\boldsymbol{\varepsilon}^{\mathbf{s}} = \{s_1, s_2 - s_1, \dots, s_n - s_{n-1}\}, \quad (36)$$

so that $\mathbf{s} \in \mathcal{A}_{\boldsymbol{\varepsilon}}^n$ if and only if $\boldsymbol{\varepsilon}^{\mathbf{s}} = \boldsymbol{\varepsilon}$.

Suppose that among $\boldsymbol{\varepsilon}$ are l distinct numbers, appearing with multiplicities m_1, \dots, m_l , necessarily summing to n . Then letting $f_{\boldsymbol{\varepsilon}}^n(\mathbf{s})$ be the probability mass function of (S_1, \dots, S_n) as given by (34), we have

$$|\mathcal{A}_{\boldsymbol{\varepsilon}}^n| = \frac{n!}{m_1! m_2! \dots m_l!} \quad \text{and} \quad f_{\boldsymbol{\varepsilon}}^n(\mathbf{s}) = \frac{1}{|\mathcal{A}_{\boldsymbol{\varepsilon}}^n|} \mathbb{1}(\mathbf{s} \in \mathcal{A}_{\boldsymbol{\varepsilon}}^n) = \frac{1}{|\mathcal{A}_{\boldsymbol{\varepsilon}}^n|} \mathbb{1}(\boldsymbol{\varepsilon}^{\mathbf{s}} = \boldsymbol{\varepsilon}), \quad (37)$$

that is, the distribution $f_{\boldsymbol{\varepsilon}}^n(\mathbf{s})$ is uniform over $\mathcal{A}_{\boldsymbol{\varepsilon}}^n$.

The following result is a conditional version of Theorem 1.4.

Theorem 3.1. *Let $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n$ be arbitrary elements of a finite set $\mathcal{A} \subset \mathbb{R}$, not necessarily distinct, π a uniform random permutation of $\{1, 2, \dots, n\}$, S_k and W_k as in (34), and $\gamma^2 = n^{-1} \sum_{i=1}^n \varepsilon_i^2$. Then there exists a positive universal constant C , and for every $\nu > 0$ positive constants K_1, K_2 and λ_0 depending only on \mathcal{A} and ν such that for any integer $n \geq 1$ and every $\eta \geq \nu$ one may construct a version of $(W_k)_{0 \leq k \leq n}$ and Gaussian random variables $(Z_k)_{0 \leq k \leq n}$ with zero mean and covariance*

$$\text{Cov}(Z_i, Z_j) = \frac{(i \wedge j)(n - (i \vee j))}{n} \quad (38)$$

on the same probability space such that

$$\begin{aligned} & \mathbb{E} \exp(\lambda \max_{0 \leq i \leq n} |W_i - \eta Z_i|) \\ & \leq \exp \left(C \log n + \frac{K_1 \lambda^2 S_n^2}{n} + K_2 \lambda^2 n (\gamma^2 - \eta^2)^2 \right) \quad \text{for any } \lambda \leq \lambda_0. \end{aligned} \quad (39)$$

Proof. As the result holds trivially for $\lambda \leq 0$ we need consider only $\lambda > 0$. Also, as $W_0 = 0$ and $Z_0 = 0$ by convention it suffices to consider the maximum over $1 \leq i \leq n$ in (39). We use Theorem 2.2 and induction to prove the theorem.

Recall the constants α_1 from Lemma 2.9 depending only on \mathcal{A} , and c_1, c_2 and θ_2 from Theorem 2.2, depending only on \mathcal{A} and ν . With B given in (17), letting θ_5 be the unique positive solution to

$$\frac{1}{\sqrt{1 - B^4 \theta^2/2}} = \frac{4}{3}, \quad (40)$$

depending only on \mathcal{A} . We will demonstrate the claim holds with

$$C = \frac{2 + \log 4}{\log(3/2)}, \quad K_1 = 8c_1, \quad K_2 = 18c_2 \quad \text{and} \quad \lambda_0 = \sqrt{\frac{\alpha_1}{32c_1}} \wedge \frac{\theta_2}{2} \wedge \frac{\theta_5}{\sqrt{72c_2}}. \quad (41)$$

Note that any multiset $\boldsymbol{\varepsilon} = \{\varepsilon_1, \dots, \varepsilon_n\}$ of elements of \mathcal{A} lies in exactly one set of the form

$$\mathcal{B}^n(a, b) = \{\{\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n\} : \sum_{i=1}^n \varepsilon_i = a, \frac{1}{n} \sum_{i=1}^n \varepsilon_i^2 = b^2\}$$

as a and b range over all pairs of feasible values of S_n and γ , respectively. Fix one such feasible pair a, b , which may be notationally suppressed when clear from context, let $\boldsymbol{\varepsilon} \in \mathcal{B}^n(a, b)$ be arbitrary and fix any value $\eta > 0$.

With $f_{\boldsymbol{\varepsilon}}^n(\mathbf{s})$ the probability mass function of (S_1, \dots, S_n) given in (37) and $\phi^n(\mathbf{z})$ the probability density function of a Gaussian random vector (Z_1, \dots, Z_n) with mean zero and covariance (38), we show that for each $n \geq 1$, we can construct a joint probability function $\rho_{\boldsymbol{\varepsilon}}^n(\mathbf{s}, \mathbf{z})$ on $\mathcal{A}_{\boldsymbol{\varepsilon}}^n \times \mathbb{R}^n$ having the desired marginals

$$\sum_{\mathbf{s} \in \mathcal{A}_{\boldsymbol{\varepsilon}}^n} \rho_{\boldsymbol{\varepsilon}}^n(\mathbf{s}, \mathbf{z}) = \phi^n(\mathbf{z}) \quad \text{and} \quad \int_{\mathbb{R}^n} \rho_{\boldsymbol{\varepsilon}}^n(\mathbf{s}, \mathbf{z}) d\mathbf{z} = f_{\boldsymbol{\varepsilon}}^n(\mathbf{s}) \quad (42)$$

and satisfying the exponential bound

$$\begin{aligned} \int_{\mathbb{R}^n} \sum_{\mathbf{s} \in \mathcal{A}_{\boldsymbol{\varepsilon}}^n} \left[\exp \left(\lambda \max_{1 \leq i \leq n} \left| s_i - \frac{ia}{n} - \eta z_i \right| \right) \rho_{\boldsymbol{\varepsilon}}^n(\mathbf{s}, \mathbf{z}) \right] d\mathbf{z} \\ \leq \exp \left(C \log n + \frac{K_1 \lambda^2 a^2}{n} + K_2 \lambda^2 n (b^2 - \eta^2)^2 \right) \quad \text{for all } \lambda \in (0, \lambda_0], \end{aligned} \quad (43)$$

for all $\eta \geq \nu$, with C, K_1, K_2 and λ_0 as in (41), with C universal and the latter three constants depending only on \mathcal{A} and ν .

We will prove the claim by induction on n . For $n = 1$ we note that $W_1 = 0$ by (2) and $Z_1 = 0$ by convention, since it has mean zero and covariance given by (38). Hence (39) holds for $n = 1$ for all C , all nonnegative K_1, K_2 , and all λ_0 , and in particular for the set of constants specified in (41).

Given $n \geq 2$, suppose that for all $l = 1, 2, \dots, n-1$ and all multisubsets $\boldsymbol{\zeta}$ of \mathcal{A} of size l we can construct $\rho_{\boldsymbol{\zeta}}^l(\mathbf{s}, \mathbf{z})$ satisfying (42) and (43). Take $k = \lfloor n/2 \rfloor$, let \sqcup denote multiset union and define the sets

$$\mathcal{S}_{\boldsymbol{\varepsilon}}^{n,k} = \{s : \sum_{\varepsilon \in \boldsymbol{\varepsilon}_1} \varepsilon = s \text{ for some } \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \text{ such that } |\boldsymbol{\varepsilon}_1| = k, \boldsymbol{\varepsilon}_1 \sqcup \boldsymbol{\varepsilon}_2 = \boldsymbol{\varepsilon}\}, \quad \text{and}$$

$$\mathcal{B}_{\boldsymbol{\varepsilon}}^{n,k}(s) = \{(\boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2) : \sum_{\varepsilon \in \boldsymbol{\varepsilon}_1} \varepsilon = s, |\boldsymbol{\varepsilon}_1| = k, \boldsymbol{\varepsilon}_1 \sqcup \boldsymbol{\varepsilon}_2 = \boldsymbol{\varepsilon}\} \quad \text{for } s \in \mathcal{S}_{\boldsymbol{\varepsilon}}^{n,k}.$$

That is, $\mathcal{S}_{\boldsymbol{\epsilon}}^{n,k}$ is the set of all feasible values at time k of a path having increments $\boldsymbol{\epsilon}$, and $\mathcal{B}_{\boldsymbol{\epsilon}}^{n,k}(s)$ is the set of all ways of dividing the n increments $\boldsymbol{\epsilon}$ into sets of sizes k and $n-k$ so that the path at time k takes the value s . Counting the number of paths that take the value $s \in \mathcal{S}_{\boldsymbol{\epsilon}}^{n,k}$ at time k shows that $g_{\boldsymbol{\epsilon}}^{n,k}(s)$, the marginal density of S_k in $f_{\boldsymbol{\epsilon}}^n(\mathbf{s})$, is given by

$$g_{\boldsymbol{\epsilon}}^{n,k}(s) = \frac{\sum_{(\boldsymbol{\zeta}_1, \boldsymbol{\zeta}_2) \in \mathcal{B}_{\boldsymbol{\epsilon}}^{n,k}(s)} |\mathcal{A}_{\boldsymbol{\zeta}_1}^k| |\mathcal{A}_{\boldsymbol{\zeta}_2}^{n-k}|}{|\mathcal{A}_{\boldsymbol{\epsilon}}^n|}. \quad (44)$$

Similarly, let $h^{n,k}(z)$ denote the marginal density function of Z_k in $\phi^n(\mathbf{z})$, that of the Gaussian distribution with mean zero and variance $k(n-k)/n$. By Theorem 2.2, there exists a joint density function $\psi_{\boldsymbol{\epsilon}}^{n,k}(s, z)$ on $\mathcal{S}_{\boldsymbol{\epsilon}}^{n,k} \times \mathbb{R}$ and positive constants c_1, c_2 and θ_2 , depending only on \mathcal{A} and ν , such that

$$\int \psi_{\boldsymbol{\epsilon}}^{n,k}(s, z) dz = g_{\boldsymbol{\epsilon}}^{n,k}(s), \quad \sum_{s \in \mathcal{S}_{\boldsymbol{\epsilon}}^{n,k}} \psi_{\boldsymbol{\epsilon}}^{n,k}(s, z) = h^{n,k}(z), \quad (45)$$

and for $\theta \leq \theta_2$ and $\eta \geq \nu$,

$$\int \sum_{s \in \mathcal{S}_{\boldsymbol{\epsilon}}^{n,k}} \left[\exp \left(\theta \left| s - \frac{ka}{n} - \eta z \right| \right) \psi_{\boldsymbol{\epsilon}}^{n,k}(s, z) \right] dz \leq \exp \left(3 + \frac{c_1 \theta^2 a^2}{n} + c_2 \theta^2 n (b^2 - \eta^2)^2 \right). \quad (46)$$

For $s \in \mathcal{S}_{\boldsymbol{\epsilon}}^{n,k}, z \in \mathbb{R}$, and recalling the definition (36) of $\boldsymbol{\epsilon}^s, \mathbf{s}^1, \mathbf{s}^2$ such that $(\boldsymbol{\epsilon}^{s^1}, \boldsymbol{\epsilon}^{s^2}) \in \mathcal{B}_{\boldsymbol{\epsilon}}^{n,k}(s)$, $\mathbf{z}^1 \in \mathbb{R}^k$ and $\mathbf{z}^2 \in \mathbb{R}^{n-k}$, let

$$\gamma_{\boldsymbol{\epsilon}}^n(s, z, \mathbf{s}^1, \mathbf{z}^1, \mathbf{s}^2, \mathbf{z}^2) = \psi_{\boldsymbol{\epsilon}}^{n,k}(s, z) P_{\boldsymbol{\epsilon}, s}(\boldsymbol{\epsilon}^{s^1}, \boldsymbol{\epsilon}^{s^2}) \rho_{\boldsymbol{\epsilon}^{s^1}}^k(\mathbf{s}^1, \mathbf{z}^1) \rho_{\boldsymbol{\epsilon}^{s^2}}^{n-k}(\mathbf{s}^2, \mathbf{z}^2) \quad (47)$$

where

$$P_{\boldsymbol{\epsilon}, s}(\boldsymbol{\epsilon}_1, \boldsymbol{\epsilon}_2) = \frac{|\mathcal{A}_{\boldsymbol{\epsilon}_1}^k| |\mathcal{A}_{\boldsymbol{\epsilon}_2}^{n-k}|}{\sum_{(\boldsymbol{\zeta}_1, \boldsymbol{\zeta}_2) \in \mathcal{B}_{\boldsymbol{\epsilon}}^{n,k}(s)} |\mathcal{A}_{\boldsymbol{\zeta}_1}^k| |\mathcal{A}_{\boldsymbol{\zeta}_2}^{n-k}|} \mathbb{1}((\boldsymbol{\epsilon}_1, \boldsymbol{\epsilon}_2) \in \mathcal{B}_{\boldsymbol{\epsilon}}^{n,k}(s)).$$

Interpreting (47) in terms of a construction, one first samples the joint values s and z of the coupled random walk and Gaussian path at time k , then chooses increments corresponding to \mathbf{s}^1 and \mathbf{s}^2 , the first and last half of the walk according to their likelihood over the choices of those whose increments over the first half of the walk sum to s , and whose union of increments over both halves must be $\boldsymbol{\epsilon}$, and then samples coupled values of the paths with discrete Brownian bridges before and after time k .

One may verify that $\gamma_{\boldsymbol{\epsilon}}^n$ is a density function by integrating over \mathbf{z}^1 and \mathbf{z}^2 using the second equality in (42) followed by applying the second equality in (37), integrating over z , and then summing over all \mathbf{s}^1 and \mathbf{s}^2 and s , this last operation being equivalent to summing over all paths \mathbf{s} with increments $\boldsymbol{\epsilon}$, see (50) and the explanation following.

Now, let $(S, Z, \mathbf{S}^1, \mathbf{Z}^1, \mathbf{S}^2, \mathbf{Z}^2)$ be a random vector with density $\gamma_{\boldsymbol{\epsilon}}^n$ where $\mathbf{S}^1 = (S_i^1)_{1 \leq i \leq k}$, $\mathbf{S}^2 = (S_i^2)_{1 \leq i \leq n-k}$ and $\mathbf{Z}^1 = (Z_i^1)_{1 \leq i \leq k}$, $\mathbf{Z}^2 = (Z_i^2)_{1 \leq i \leq n-k}$. Let \mathbf{S} be obtained by ‘piecing’ the paths \mathbf{S}^1 and \mathbf{S}^2 together at time k according to the rule

$$S_i = \begin{cases} S_i^1 & 1 \leq i \leq k \\ S + S_{i-k}^2 & k < i \leq n, \end{cases} \quad (48)$$

here noting $S_k = S$, and define \mathbf{Z} by

$$Z_i = \begin{cases} Z_i^1 + \frac{i}{k}Z & 1 \leq i \leq k \\ Z_{i-k}^2 + \frac{n-i}{n-k}Z & k < i \leq n, \end{cases} \quad (49)$$

here noting likewise that $Z_k = Z$, since $Z_k^1 = 0$. Now as in [6], we demonstrate that $\rho_{\boldsymbol{\epsilon}}^n(\mathbf{s}, \mathbf{z})$, the joint density of (\mathbf{S}, \mathbf{Z}) , achieves the desired marginals (42) and exponential bound (43).

1. *Marginal distribution of \mathbf{S} .* Let \mathbf{s} be the path constructed from s, \mathbf{s}^1 and \mathbf{s}^2 as \mathbf{S} is constructed from S, \mathbf{S}^1 and \mathbf{S}^2 in (48). Note that

$$\{\mathbf{s} : \mathbf{s} \in \mathcal{A}_{\boldsymbol{\epsilon}}^n\} = \{\mathbf{s} : (\boldsymbol{\epsilon}^{\mathbf{s}^1}, \boldsymbol{\epsilon}^{\mathbf{s}^2}) \in \mathcal{B}_{\boldsymbol{\epsilon}}^{n,k}(s_k)\},$$

and that $S_k = S$ almost surely. Hence, if $\mathbf{S} \notin \mathcal{A}_{\boldsymbol{\epsilon}}^n$ then from (47) \mathbf{S} has probability zero. For the marginal of $\gamma_{\boldsymbol{\epsilon}}^n$ to be non-zero on $s, \mathbf{s}^1, \mathbf{s}^2$, first s must be a feasible value at time k for a path with increments $\boldsymbol{\epsilon}$, then \mathbf{s}^1 must be a path of increments that attains the value s at time k , and finally the collection of increments determined by \mathbf{s}^1 and \mathbf{s}^2 must match the given set $\boldsymbol{\epsilon}$ of increments. In this case we obtain from (42), (45) and (44), that the marginal distribution of $(S, \mathbf{S}^1, \mathbf{S}^2)$ is given by

$$\begin{aligned} & \int \gamma_{\boldsymbol{\epsilon}}^n(s, z, \mathbf{s}^1, \mathbf{z}^1, \mathbf{s}^2, \mathbf{z}^2) d\mathbf{z}^2 d\mathbf{z}^1 dz \\ &= g_{\boldsymbol{\epsilon}}^{n,k}(s) P_{\boldsymbol{\epsilon},s}(\boldsymbol{\epsilon}^{\mathbf{s}^1}, \boldsymbol{\epsilon}^{\mathbf{s}^2}) f_{\boldsymbol{\epsilon}^{\mathbf{s}^1}}^k(\mathbf{s}^1) f_{\boldsymbol{\epsilon}^{\mathbf{s}^2}}^{n-k}(\mathbf{s}^2) \\ &= \frac{\sum_{(\boldsymbol{\zeta}_1, \boldsymbol{\zeta}_2) \in \mathcal{B}_{\boldsymbol{\epsilon}}^{n,k}(s)} |\mathcal{A}_{\boldsymbol{\zeta}_1}^k| |\mathcal{A}_{\boldsymbol{\zeta}_2}^{n-k}|}{|\mathcal{A}_{\boldsymbol{\epsilon}}^n|} \frac{|\mathcal{A}_{\boldsymbol{\epsilon}^{\mathbf{s}^1}}^k| |\mathcal{A}_{\boldsymbol{\epsilon}^{\mathbf{s}^2}}^{n-k}|}{\sum_{(\boldsymbol{\zeta}_1, \boldsymbol{\zeta}_2) \in \mathcal{B}_{\boldsymbol{\epsilon}}^{n,k}(s)} |\mathcal{A}_{\boldsymbol{\zeta}_1}^k| |\mathcal{A}_{\boldsymbol{\zeta}_2}^{n-k}|} \frac{1}{|\mathcal{A}_{\boldsymbol{\epsilon}^{\mathbf{s}^1}}^k| |\mathcal{A}_{\boldsymbol{\epsilon}^{\mathbf{s}^2}}^{n-k}|} \\ &= \frac{1}{|\mathcal{A}_{\boldsymbol{\epsilon}}^n|} \\ &= f_{\boldsymbol{\epsilon}}^n(\mathbf{s}). \end{aligned} \quad (50)$$

Now observing that (48) gives a one-to-one correspondence between $(S, \mathbf{S}^1, \mathbf{S}^2)$ and \mathbf{S} we find that \mathbf{S} has marginal density $f_{\boldsymbol{\epsilon}}^n(\mathbf{s})$ as in (37).

2. *Marginal distribution of \mathbf{Z} .* Consider $\mathcal{A}_{\boldsymbol{\epsilon}_1}^k \times \mathcal{A}_{\boldsymbol{\epsilon}_2}^{n-k}$, the set of all pairs of paths $(\mathbf{s}^1, \mathbf{s}^2)$ with increments $\boldsymbol{\epsilon}_1$ and $\boldsymbol{\epsilon}_2$ respectively. Using (42) and (45), and noting that $(\boldsymbol{\epsilon}^{\mathbf{s}^1}, \boldsymbol{\epsilon}^{\mathbf{s}^2}) = (\boldsymbol{\epsilon}_1, \boldsymbol{\epsilon}_2)$ for

$(\mathbf{s}^1, \mathbf{s}^2) \in \mathcal{A}_{\boldsymbol{\epsilon}_1}^k \times \mathcal{A}_{\boldsymbol{\epsilon}_2}^{n-k}$, the marginal distribution of $Z, \mathbf{Z}^1, \mathbf{Z}^2$ is given by

$$\begin{aligned}
& \sum_{s \in \mathcal{S}_{\boldsymbol{\epsilon}}^{n,k}} \sum_{(\boldsymbol{\epsilon}_1, \boldsymbol{\epsilon}_2) \in \mathcal{B}_{\boldsymbol{\epsilon}}^{n,k}(s)} \sum_{(\mathbf{s}^1, \mathbf{s}^2) \in \mathcal{A}_{\boldsymbol{\epsilon}_1}^k \times \mathcal{A}_{\boldsymbol{\epsilon}_2}^{n-k}} \gamma_{\boldsymbol{\epsilon}}^n(s, z, \mathbf{s}^1, \mathbf{z}^2, \mathbf{s}^2, \mathbf{z}^2) \\
&= \sum_{s \in \mathcal{S}_{\boldsymbol{\epsilon}}^{n,k}} \psi_{\boldsymbol{\epsilon}}^{n,k}(s, z) \left[\sum_{(\boldsymbol{\epsilon}_1, \boldsymbol{\epsilon}_2) \in \mathcal{B}_{\boldsymbol{\epsilon}}^{n,k}(s)} P_{\boldsymbol{\epsilon},s}(\boldsymbol{\epsilon}_1, \boldsymbol{\epsilon}_2) \sum_{(\mathbf{s}^1, \mathbf{s}^2) \in \mathcal{A}_{\boldsymbol{\epsilon}_1}^k \times \mathcal{A}_{\boldsymbol{\epsilon}_2}^{n-k}} \rho_{\boldsymbol{\epsilon}_1}^k(\mathbf{s}^1, \mathbf{z}^1) \rho_{\boldsymbol{\epsilon}_2}^{n-k}(\mathbf{s}^2, \mathbf{z}^2) \right] \\
&= \sum_{s \in \mathcal{S}_{\boldsymbol{\epsilon}}^{n,k}} \psi_{\boldsymbol{\epsilon}}^{n,k}(s, z) \left[\sum_{(\boldsymbol{\epsilon}_1, \boldsymbol{\epsilon}_2) \in \mathcal{B}_{\boldsymbol{\epsilon}}^{n,k}(s)} P_{\boldsymbol{\epsilon},s}(\boldsymbol{\epsilon}_1, \boldsymbol{\epsilon}_2) \sum_{\mathbf{s}^1 \in \mathcal{A}_{\boldsymbol{\epsilon}_1}^k} \rho_{\boldsymbol{\epsilon}_1}^k(\mathbf{s}^1, \mathbf{z}^1) \sum_{\mathbf{s}^2 \in \mathcal{A}_{\boldsymbol{\epsilon}_2}^{n-k}} \rho_{\boldsymbol{\epsilon}_2}^{n-k}(\mathbf{s}^2, \mathbf{z}^2) \right] \\
&= \sum_{s \in \mathcal{S}_{\boldsymbol{\epsilon}}^{n,k}} \psi_{\boldsymbol{\epsilon}}^{n,k}(s, z) \left[\sum_{(\boldsymbol{\epsilon}_1, \boldsymbol{\epsilon}_2) \in \mathcal{B}_{\boldsymbol{\epsilon}}^{n,k}(s)} P_{\boldsymbol{\epsilon},s}(\boldsymbol{\epsilon}_1, \boldsymbol{\epsilon}_2) \phi^k(\mathbf{z}^1) \phi^{n-k}(\mathbf{z}^2) \right] \\
&= \phi^k(\mathbf{z}^1) \phi^{n-k}(\mathbf{z}^2) \sum_s \psi_{\boldsymbol{\epsilon}}^{n,k}(s, z) \\
&= \phi^{n-k}(\mathbf{z}^2) \phi^k(\mathbf{z}^1) h^{n,k}(z)
\end{aligned}$$

where we have used that $\sum_{(\boldsymbol{\epsilon}_1, \boldsymbol{\epsilon}_2) \in \mathcal{B}_{\boldsymbol{\epsilon}}^{n,k}(s)} P_{\boldsymbol{\epsilon},s}(\boldsymbol{\epsilon}_1, \boldsymbol{\epsilon}_2) = 1$. Hence Z, \mathbf{Z}^1 and \mathbf{Z}^2 are independent with densities $h^{n,k}(z)$, $\phi^k(\mathbf{z}^1)$ and $\phi^{n-k}(\mathbf{z}^2)$ respectively, implying that \mathbf{Z} given by (49) is a multivariate mean zero Gaussian random vector. As in [6], one can verify that \mathbf{Z} has covariances given by (38), and hence $\mathbf{Z} \sim \phi^n(\mathbf{z})$.

3. The exponential bound. For $1 \leq i \leq n$, letting

$$W_i = S_i - \frac{ia}{n},$$

we show that

$$\mathbb{E} \exp(\lambda \max_{1 \leq i \leq n} |W_i - \eta Z_i|) \leq \exp \left(C \log n + \frac{K_1 \lambda^2 a}{n} + K_2 \lambda^2 n (b^2 - \eta^2)^2 \right) \quad \text{for } \lambda \in (0, \lambda_0]$$

where C, K_1, K_2 and λ_0 are as in (41). We continue to proceed as in [6].

Again writing S for S_k , let

$$T_L := \max_{1 \leq i \leq k} \left| S_i^1 - \frac{iS}{k} - \eta Z_i^1 \right|, T_R := \max_{k < i \leq n} \left| S_{i-k}^2 - \frac{i-k}{n-k} (a - S) - \eta Z_{i-k}^2 \right|,$$

and

$$T := \left| S - \frac{ka}{n} - \eta Z \right|.$$

Note that when $1 \leq i \leq k$ we have

$$\begin{aligned}
|W_i - \eta Z_i| &= \left| S_i^1 - \frac{ia}{n} - \eta \left(Z_i^1 + \frac{iZ}{k} \right) \right| \\
&\leq \left| S_i^1 - \frac{iS}{k} - \eta Z_i^1 \right| + \left| \frac{iS}{k} - \frac{ia}{n} - \frac{i}{k} \eta Z \right| \\
&\leq T_L + \frac{i}{k} T \leq T_L + T.
\end{aligned}$$

Similarly for $k < i \leq n$ one can verify $|W_i - \eta Z_i| \leq T_R + T$, proving

$$\max_{1 \leq i \leq n} |W_i - \eta Z_i| \leq \max\{T_L + T, T_R + T\}.$$

Now fixing $\lambda \leq \lambda_0$, the inequality $\exp(x \vee y) \leq e^x + e^y$ yields

$$\exp(\lambda \max_{1 \leq i \leq n} |W_i - \eta Z_i|) \leq \exp(\lambda T_L + \lambda T) + \exp(\lambda T_R + \lambda T). \quad (51)$$

To prove that the exponential bound holds, we develop inequalities on the expectation of the two quantities on the right hand side of (51), starting with the expression involving T_L .

Note that $\boldsymbol{\varepsilon}^{s^1}$ determines S , and since $\boldsymbol{\varepsilon}$ is fixed $\boldsymbol{\varepsilon}^{s^2}$ is also determined, so by (47) the conditional density of $(\mathbf{S}^1, \mathbf{Z}^1)$ given $(\boldsymbol{\varepsilon}^{s^1}, Z)$ is $\rho_{\boldsymbol{\varepsilon}^{s^1}}^k(\mathbf{s}^1, \mathbf{z}^1)$. Now using that the moment generating functions of T_L and T are finite everywhere and that T is a function of $\{S, Z\}$, invoking the induction hypothesis and applying the Cauchy-Schwarz inequality twice, with $\gamma_1^2 = (1/k) \sum_{i=1}^k \varepsilon_{\pi(i)}^2$ we obtain

$$\begin{aligned} \mathbb{E} \exp(\lambda T_L + \lambda T) &= \mathbb{E} \left[\mathbb{E} \left(\exp(\lambda T_L) | \boldsymbol{\varepsilon}^{s^1}, Z \right) \exp(\lambda T) \right] \\ &\leq \left[\mathbb{E} \left(\mathbb{E} \left(\exp(\lambda T_L) | \boldsymbol{\varepsilon}^{s^1}, Z \right)^2 \right) \mathbb{E}(\exp(2\lambda T)) \right]^{1/2} \\ &\leq \exp(C \log k) \left[\mathbb{E} \exp \left(\frac{2K_1 \lambda^2 S^2}{k} + 2K_2 \lambda^2 k (\gamma_1^2 - \eta^2)^2 \right) \mathbb{E} \exp(2\lambda T) \right]^{1/2} \\ &\leq \exp(C \log k) \left[\mathbb{E} \exp \left(\frac{4K_1 \lambda^2 S^2}{k} \right) \mathbb{E} \exp(4K_2 \lambda^2 k (\gamma_1^2 - \eta^2)^2) \right]^{1/4} (\mathbb{E} \exp(2\lambda T))^{1/2}. \quad (52) \end{aligned}$$

For the first expectation in (52), (41) implies that $0 \leq 4K_1 \lambda^2 \leq \alpha_1$, and as $|2k - n| \leq 1$ we may invoke Lemma 2.9 to yield

$$\mathbb{E} \exp \left(\frac{4K_1 \lambda^2 S^2}{k} \right) \leq \exp \left(1 + \frac{3K_1 \lambda^2 a^2}{n} \right). \quad (53)$$

For the second expectation in (52), recalling the definition of γ_1^2 ,

$$\mathbb{E} \exp(4K_2 \lambda^2 k (\gamma_1^2 - \eta^2)^2) = \mathbb{E} \exp \left(4K_2 \lambda^2 \frac{1}{k} \left(\sum_{i=1}^k (\varepsilon_{\pi(i)}^2 - \eta^2) \right)^2 \right) = \mathbb{E} \exp \left(\theta^2 \frac{U_k^2}{k} \right), \quad (54)$$

where $\theta = 2\lambda \sqrt{K_2}$, and we write

$$U_k = \sum_{i=1}^k (\varepsilon_{\pi(i)}^2 - \eta^2) = \sum_{i=1}^n \left(\varepsilon_i^2 \mathbf{1}_{i \in \pi([k])} - \frac{k}{n} \eta^2 \right) = \sum_{i=1}^n a_i,$$

where $[k] = \{1, \dots, k\}$ so that $\pi([k]) = \{\pi(i) : i = 1, 2, \dots, k\}$, and $a_i = \varepsilon_i^2 \mathbf{1}_{i \in \pi([k])} - (k/n) \eta^2$.

To bound (54), we will argue as in Lemma 2.6. Observe that for V a standard normal random variable independent of U_k ,

$$\begin{aligned}\mathbb{E} \exp \left(\theta^2 \frac{U_k^2}{k} \right) &= \mathbb{E} \exp \left(\sqrt{2} \theta \frac{V}{\sqrt{k}} U_k \right) \\ &= \mathbb{E} \exp \left(\sqrt{2} \theta \frac{|V| \operatorname{sgn}(V)}{\sqrt{k}} U_k \right) \\ &= \mathbb{E} \exp \left(\sqrt{2} \theta \frac{|V|}{\sqrt{k}} U_k \mid \operatorname{sgn}(V) = 1 \right) P(\operatorname{sgn}(V) = 1) \\ &\quad + \mathbb{E} \exp \left(\sqrt{2} \theta \frac{|V|}{\sqrt{k}} (-U_k) \mid \operatorname{sgn}(V) = -1 \right) P(\operatorname{sgn}(V) = -1).\end{aligned}$$

Now using the independence of $|V|$ and $\operatorname{sgn}(V)$, and that $\operatorname{sgn}(V)$ is a symmetric ± 1 random variable, we obtain

$$\mathbb{E} \exp \left(\theta^2 \frac{U_k^2}{k} \right) = \frac{1}{2} \left[\mathbb{E} \exp \left(\sqrt{2} \theta U_k \frac{|V|}{\sqrt{k}} \right) + \mathbb{E} \exp \left(\sqrt{2} \theta (-U_k) \frac{|V|}{\sqrt{k}} \right) \right]. \quad (55)$$

Recall that random variables X_1, X_2, \dots, X_n are said to be negatively associated, see [15], if for any two disjoint index sets I and J ,

$$\mathbb{E}[f(X_i, i \in I)g(X_j, j \in J)] \leq \mathbb{E}[f(X_i, i \in I)]\mathbb{E}[g(X_j, j \in J)] \quad (56)$$

for all coordinatewise nondecreasing functions $f : \mathbb{R}^{|I|} \rightarrow \mathbb{R}$ and $g : \mathbb{R}^{|J|} \rightarrow \mathbb{R}$.

Let X_1, \dots, X_n be negatively associated. It is immediate that $aX_1 + b, \dots, aX_n + b$ are negatively associated for all $a \geq 0$ and $b \in \mathbb{R}$. In addition, letting $Y_i = -X_i$ for all $i = 1, \dots, n$, for f and g coordinatewise nondecreasing functions and I and J disjoint index sets, as $-f(\cdot)$ is coordinatewise nondecreasing, we have

$$\begin{aligned}\mathbb{E}[f(Y_i, i \in I)g(Y_j, j \in J)] &= \mathbb{E}[(-f(-X_i, i \in I))(-g(-X_j, j \in J))] \\ &\leq \mathbb{E}[(-f(-X_i, i \in I))]\mathbb{E}[(-g(-X_j, j \in J))] = \mathbb{E}[f(Y_i, i \in I)]\mathbb{E}[g(Y_j, j \in J)],\end{aligned}$$

demonstrating that $-X_1, \dots, -X_n$ are negatively associated. Combining these two facts, $aX_1 + b, \dots, aX_n + b$ are negatively associated for all $a \in \mathbb{R}$ and $b \in \mathbb{R}$. By a direct inductive argument on (56),

$$\mathbb{E} \left[\prod_{i=1}^n f_i(X_i) \right] \leq \prod_{i=1}^n \mathbb{E} [f_i(X_i)] \quad (57)$$

whenever the functions $f_i, i = 1, 2, \dots, n$ are all nondecreasing.

By Theorem 2.11 of [15], taking the real numbers in Definition 2.10 there to consist of k ones and $n - k$ zeros, the indicators $\mathbb{1}_{1 \in \pi([k])}, \dots, \mathbb{1}_{n \in \pi([k])}$ are negatively associated; hence so are a_1, \dots, a_n and $-a_1, \dots, -a_n$. Thus, by (57), we have

$$\begin{aligned}\mathbb{E} \left[\exp \left(\sqrt{2} \theta U_k \frac{|V|}{\sqrt{k}} \right) \mid V \right] &= \mathbb{E} \left[\exp \left(\sqrt{2} \theta \sum_{i=1}^n a_i \frac{|V|}{\sqrt{k}} \right) \mid V \right] \\ &\leq \prod_{i=1}^n \mathbb{E} \left[\exp \left(\sqrt{2} \theta a_i \frac{|V|}{\sqrt{k}} \right) \mid V \right] = \prod_{i=1}^k \mathbb{E} \left[\exp \left(\sqrt{2} \theta \left(\varepsilon_{\pi(i)}^2 - \eta^2 \right) \frac{|V|}{\sqrt{k}} \right) \mid V \right]. \quad (58)\end{aligned}$$

Now since $-\eta^2 \leq \varepsilon_{\pi(i)}^2 - \eta^2 \leq B^2 - \eta^2$, using Hoeffding's lemma (12) with $\mu = b^2 - \eta^2$, the mean of $\varepsilon_{\pi(i)}^2 - \eta^2$, we obtain

$$\begin{aligned} \prod_{i=1}^k \mathbb{E} \left[\exp \left(\sqrt{2}\theta (\varepsilon_{\pi(i)}^2 - \eta^2) \frac{|V|}{\sqrt{k}} \right) \middle| V \right] &\leq \exp \left(\frac{B^4 \theta^2 V^2}{4k} + \sqrt{2}\theta \mu \frac{|V|}{\sqrt{k}} \right)^k \\ &= \exp \left(\frac{B^4 \theta^2 V^2}{4} + \sqrt{2}\theta \mu \sqrt{k}|V| \right) \\ &\leq \exp \left(\frac{B^4 \theta^2 V^2}{4} + \sqrt{2}\theta \mu \sqrt{k}V \right) + \exp \left(\frac{B^4 \theta^2 V^2}{4} + \sqrt{2}\theta \mu \sqrt{k}(-V) \right). \end{aligned}$$

Using that V and $-V$ have the same distribution, taking expectation in (58) and then applying the non-central chi square identity (13) yields

$$\begin{aligned} \mathbb{E} \left[\exp \left(\sqrt{2}\theta U_k \frac{|V|}{\sqrt{k}} \right) \right] &\leq 2\mathbb{E} \left[\exp \left(\frac{B^4 \theta^2 V^2}{4} + \sqrt{2}\theta \mu \sqrt{k}V \right) \right] \\ &= \frac{2}{\sqrt{1 - B^4 \theta^2/2}} \exp \left(\frac{k\theta^2 \mu^2}{\sqrt{1 - B^4 \theta^2/2}} \right) \leq \frac{8}{3} \exp \left(\frac{4}{3} k\theta^2 \mu^2 \right) \quad (59) \end{aligned}$$

for all $0 \leq \theta \leq \theta_5$, by (40).

Using the fact that $-a_1, \dots, -a_n$ are negatively associated and that $-a_i$ and a_i have supports over intervals of equal length for all $i = 1, 2, \dots, n$, (59) holds with U_k replaced by $-U_k$. Thus, by (55),

$$\mathbb{E} \exp \left(\theta^2 \frac{U_k^2}{k} \right) \leq \frac{8}{3} \exp \left(\frac{4}{3} k\theta^2 \mu^2 \right) \quad \text{for } 0 \leq \theta \leq \theta_5. \quad (60)$$

Using (41) we see that $0 \leq 4K_2\lambda^2 \leq \theta_5^2$, and as $k \leq \frac{2n}{3}$, by (54) and (60), and recalling that $\mu = b^2 - \eta^2$, we have

$$\begin{aligned} \mathbb{E} \exp(4K_2\lambda^2 k(\gamma_1^2 - \eta^2)^2) &\leq \frac{8}{3} \exp \left(\frac{16}{3} K_2\lambda^2 k(b^2 - \eta^2)^2 \right) \leq 3 \exp \left(\frac{32}{9} K_2\lambda^2 n(b^2 - \eta^2)^2 \right). \quad (61) \end{aligned}$$

For the third expectation in (52), again by (41), $0 \leq 2\lambda \leq \theta_2$. Hence by (46),

$$\mathbb{E} \exp(2\lambda T) \leq \exp \left(3 + \frac{4c_1\lambda^2 a^2}{n} + 4c_2\lambda^2 n(b^2 - \eta^2)^2 \right). \quad (62)$$

Applying bounds (53), (61) and (62) in (52), and setting

$$Q_{12} = 1 + \frac{3K_1\lambda^2 a^2}{n} + \frac{32K_2\lambda^2 n(b^2 - \eta^2)^2}{9} \quad \text{and} \quad Q_3 = 3 + \frac{4c_1\lambda^2 a^2}{n} + 4c_2\lambda^2 n(b^2 - \eta^2)^2,$$

we obtain

$$\begin{aligned} \mathbb{E} \exp(\lambda T_L + \lambda T) &\leq 3^{1/4} \exp \left(C \log k + \frac{1}{4} Q_{12} + \frac{1}{2} Q_3 \right) \\ &\leq 2 \exp \left(C \log k + 2 + \frac{(3K_1 + 8c_1)\lambda^2 a^2}{4n} + \frac{(8K_2 + 18c_2)}{9} \lambda^2 n(b^2 - \eta^2)^2 \right). \end{aligned}$$

Again by (41), $3K_1 + 8c_1 = 4K_1$ and $8K_2 + 18c_2 = 9K_2$. Since $k \leq 2n/3$, we have

$$\log k = \log n - \log(n/k) \leq \log n - \log(3/2).$$

Thus, using from (41) that $C \log(3/2) = \log 4 + 2$,

$$\begin{aligned} \mathbb{E} \exp(\lambda T_L + \lambda T) &\leq 2 \exp \left(C \log n - C \log(3/2) + 2 + \frac{K_1 \lambda^2 a^2}{n} + K_2 \lambda^2 n (b^2 - \eta^2)^2 \right) \\ &= \frac{1}{2} \exp \left(C \log n + \frac{K_1 \lambda^2 a^2}{n} + K_2 \lambda^2 n (b^2 - \eta^2)^2 \right). \end{aligned}$$

In like manner we obtain this same bound on $\mathbb{E} \exp(\lambda T_R + \lambda T)$, so (51), now yields

$$\exp(\lambda \max_{1 \leq i \leq n} |W_i - \eta Z_i|) \leq \exp \left(C \log n + \frac{K_1 \lambda^2 a^2}{n} + K_2 \lambda^2 n (b^2 - \eta^2)^2 \right).$$

This step completes the induction, and the proof. \square

Proof of Theorem 1.4: Let \mathcal{A} be the set of the r distinct values $\{a_1, \dots, a_r\}$ and let $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n$ be exchangeable random variables taking values in \mathcal{A} . Let

$$\mathbf{M} = (M_1, \dots, M_r) \quad \text{where for } j = 1, \dots, r \text{ we set } M_j = \sum_{i=1}^n \mathbb{1}(\varepsilon_i = a_j),$$

the number of components of the multiset $\boldsymbol{\varepsilon} = \{\varepsilon_1, \dots, \varepsilon_n\}$ that take on the value a_j . With \mathcal{L} denoting distribution, or law, clearly

$$\mathcal{L}(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n) = \sum_{\mathbf{m} \geq 0} \mathcal{L}(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n | \mathbf{M} = \mathbf{m}) P(\mathbf{M} = \mathbf{m})$$

where $\mathbf{m} = (m_1, \dots, m_r)$ and $\mathbf{m} \geq 0$ is to be interpreted componentwise. As \mathbf{M} is a symmetric function of $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n$, the conditional law $\mathcal{L}(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n | \mathbf{M} = \mathbf{m})$ inherits exchangeability from $\mathcal{L}(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n)$, that is,

$$\mathcal{L}(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n | \mathbf{M} = \mathbf{m}) =_d \mathcal{L}(\varepsilon_{\pi(1)}, \varepsilon_{\pi(2)}, \dots, \varepsilon_{\pi(n)} | \mathbf{M} = \mathbf{m})$$

where π is uniformly chosen from \mathcal{P}_n . In particular, given $\mathbf{M} = \mathbf{m}$,

$$\sum_{i=1}^k \varepsilon_i =_d \sum_{i=1}^k \varepsilon_{\pi(i)} \quad \text{for all } k = 1, \dots, n$$

where $=_d$ denotes equality in distribution. Hence, (39) of Theorem 3.1 yields the version of the first claim of Theorem 1.4 when conditioning on \mathbf{M} , and taking expectation over \mathbf{M} yields that result.

We now demonstrate the second claim under the assumption that $0 \notin \mathcal{A}$, which together with \mathcal{A} finite implies that

$$v = \min_{a \in \mathcal{A}} |a| \tag{63}$$

is positive. With this value of v the constants c_1, c_2 and θ_2 as given by Theorem 2.2 depend only on \mathcal{A} , and let C, K_1, K_2 and λ_0 be as given in (41) for this v . As $\gamma \geq v$, conditional on $\varepsilon_1, \dots, \varepsilon_n$,

inequality (39) of Theorem 3.1 holds for $\eta = \gamma$, and the argument is completed by taking expectation over \mathbf{M} as for the proof of the first claim.

For the last claim, under the hypotheses that $\varepsilon_1, \dots, \varepsilon_n$ are i.i.d. mean zero random variables, since K_1 depends only on \mathcal{A} , by Lemma 2.6 there exists $\lambda > 0$ depending only on \mathcal{A} such that

$$\mathbb{E} \left(\frac{K_1 \lambda^2 S_n^2}{n} \right) \leq 2.$$

Thus from the second claim of the theorem we obtain

$$\mathbb{E} \exp(\lambda \max_{0 \leq k \leq n} |W_k - \sqrt{n} \gamma B_{k/n}|) \leq 2 \exp(C \log n),$$

and applying Markov's inequality yields

$$\begin{aligned} P \left(\max_{0 \leq k \leq n} |W_k - \sqrt{n} \gamma B_{k/n}| \geq \lambda^{-1} C \log n + x \right) &\leq \frac{\mathbb{E} \exp(\lambda \max_{0 \leq k \leq n} |W_k - \sqrt{n} \gamma B_{k/n}|) e^{-\lambda x}}{\exp(C \log n)} \\ &\leq \frac{2 \exp(C \log n)}{\exp(C \log n)} e^{-\lambda x} = 2e^{-\lambda x}. \end{aligned}$$

□

4 Proof of Theorem 1.3

In this final section we prove Theorem 1.3 by first demonstrating a ‘finite n version’ of the desired result in the following lemma.

Lemma 4.1. *There exists a constant A such that for every finite set \mathcal{A} of real numbers not containing zero, there exists a constant $\lambda > 0$ such that for any positive integer n , any $\varepsilon, \varepsilon_1, \varepsilon_2, \dots, \varepsilon_n$ i.i.d. random variables with mean zero and variance one satisfying $\mathbb{E} \varepsilon^3 = 0$ and taking values in \mathcal{A} , and $S_k = \sum_{i=1}^k \varepsilon_i, k = 1, \dots, n$, it is possible to construct a version of the sequence $(S_k)_{0 \leq k \leq n}$ and Gaussian random variables $(Z_k)_{0 \leq k \leq n}$ with mean zero and $\text{Cov}(Z_i, Z_j) = i \wedge j$ on the same probability space such that*

$$\mathbb{E} \exp(\lambda |S_n - Z_n|) \leq A \tag{64}$$

and

$$\mathbb{E} \exp(\lambda \max_{0 \leq k \leq n} |S_k - Z_k|) \leq A \exp(A \log n). \tag{65}$$

Proof. As in Theorem 3.1 it suffices to prove the result with the maximum taken over $1 \leq k \leq n$. Recall the positive constant θ_1 from Theorem 2.1, the values $\vartheta_{\ell(X)}$ from Lemma 2.6, B from (17), and let C, K_1, K_2 and λ_0 be as in Theorem 1.4 for $v = \min_{a \in \mathcal{A}} |a|$. Set

$$\lambda = \min \left\{ \frac{\theta_1}{2}, \frac{\lambda_0}{4}, \frac{\vartheta_{\ell(\varepsilon)}}{4\sqrt{K_1}}, \frac{\vartheta_{\ell(\varepsilon^2)}}{\sqrt{2}}, \frac{1}{B+1} \right\}. \tag{66}$$

Let $g^n(s)$ and $h^n(z)$ denote the mass function of S_n and the density of Z_n respectively; in particular $h^n(z)$ is just the $\mathcal{N}(0, n)$ density. By Theorem 2.1, as $2\lambda \leq \theta_1$, with \mathcal{S}^n the support of S_n , there is a joint probability function $\psi^n(s, z)$ on $\mathcal{S}^n \times \mathbb{R}$ such that

$$\int_{\mathbb{R}} \psi^n(s, z) dz = g^n(s), \quad \sum_{s \in \mathcal{S}^n} \psi^n(s, z) = h^n(z), \tag{67}$$

and

$$\int_{\mathbb{R}} \left[\sum_{s \in \mathcal{S}^n} \exp(2\lambda|s-z|) \psi^n(s, z) \right] dz \leq 8. \quad (68)$$

Given any multiset of values $\boldsymbol{\varepsilon} = \{\varepsilon_1, \dots, \varepsilon_n\}$ from \mathcal{A} , let $\rho_{\boldsymbol{\varepsilon}}^n(\mathbf{s}, \mathbf{z})$ be the joint density function guaranteed by Theorem 3.1; from that result, the marginal distributions of \mathbf{s} and \mathbf{z} are, respectively, $f_{\boldsymbol{\varepsilon}}^n(\mathbf{s})$ as in (37), and $\phi^n(\mathbf{z})$, that of a mean zero Gaussian vector with covariance (38).

For any $s \in \mathcal{S}^n$, define

$$\mathcal{B}^n(s) = \{ \{ \varepsilon_1, \varepsilon_2, \dots, \varepsilon_n \} : \sum_{i=1}^n \varepsilon_i = s \}.$$

Now, recalling the definition (36) of $\boldsymbol{\varepsilon}^s$, for $s \in \mathcal{S}^n$, \mathbf{s} such that $\boldsymbol{\varepsilon}^s \in \mathcal{B}^n(s)$, $z \in \mathbb{R}$ and $\tilde{\mathbf{z}} \in \mathbb{R}^n$, let

$$\gamma^n(s, z, \mathbf{s}, \tilde{\mathbf{z}}) = \psi^n(s, z) P(\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^s | S_n = s) \rho_{\boldsymbol{\varepsilon}^s}^n(\mathbf{s}, \tilde{\mathbf{z}}), \quad (69)$$

where the multiset $\boldsymbol{\varepsilon}$ on the right hand side is composed of n independent random variables distributed as ε . Interpreting (69) in terms of a construction, to obtain $(S, Z, \mathbf{S}, \tilde{\mathbf{Z}})$ one first samples the joint values S and Z of the coupled random walk and Gaussian path at time n , then conditional on the terminal value S , one samples increments $\boldsymbol{\varepsilon}$ consistent with the path \mathbf{s} from their i.i.d. distribution, and finally one couples a walk \mathbf{S} to the discrete Brownian bridge $\tilde{\mathbf{Z}}$ in such a way that a certain multiple of it and (W_1, \dots, W_n) given by

$$W_i = S_i - \frac{i}{n} S_n \quad (70)$$

are close.

To verify that (69) determines a probability function, recalling (35), note first that

$$\begin{aligned} & \sum_{\mathbf{s}: \boldsymbol{\varepsilon}^s \in \mathcal{B}^n(s)} P(\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^s | S_n = s) \rho_{\boldsymbol{\varepsilon}^s}^n(\mathbf{s}, \tilde{\mathbf{z}}) \\ &= \sum_{\boldsymbol{\delta} \in \mathcal{B}^n(s)} \sum_{\mathbf{s} \in \mathcal{A}_{\boldsymbol{\delta}}^n} P(\boldsymbol{\varepsilon} = \boldsymbol{\delta} | S_n = s) \rho_{\boldsymbol{\delta}}^n(\mathbf{s}, \tilde{\mathbf{z}}) = \sum_{\boldsymbol{\delta} \in \mathcal{B}^n(s)} P(\boldsymbol{\varepsilon} = \boldsymbol{\delta} | S_n = s) \sum_{\mathbf{s} \in \mathcal{A}_{\boldsymbol{\delta}}^n} \rho_{\boldsymbol{\delta}}^n(\mathbf{s}, \tilde{\mathbf{z}}) \\ &= \sum_{\boldsymbol{\delta} \in \mathcal{B}^n(s)} P(\boldsymbol{\varepsilon} = \boldsymbol{\delta} | S_n = s) \phi^n(\tilde{\mathbf{z}}) = \phi^n(\tilde{\mathbf{z}}). \end{aligned}$$

Now by (67),

$$\sum_{s \in \mathcal{S}^n} \sum_{\mathbf{s}: \boldsymbol{\varepsilon}^s \in \mathcal{B}^n(s)} \gamma^n(s, z, \mathbf{s}, \tilde{\mathbf{z}}) = h^n(z) \phi^n(\tilde{\mathbf{z}}), \quad (71)$$

and integrating over z and $\tilde{\mathbf{z}}$ yields 1.

Let $(S, Z, \mathbf{S}, \tilde{\mathbf{Z}})$ be a random vector sampled from $\gamma^n(s, z, \mathbf{s}, \tilde{\mathbf{z}})$, and define $\mathbf{Z} = (Z_1, \dots, Z_n)$ by

$$Z_i = \tilde{Z}_i + \frac{i}{n} Z.$$

Using that Z and $\tilde{\mathbf{Z}}$ are independent by (71), and that the latter has covariance given by (38), it follows that \mathbf{Z} is a mean zero Gaussian random vector with $\text{Cov}(Z_i, Z_j) = i \wedge j$.

Regarding the marginals of \mathbf{s} , integrating (69) over z and $\tilde{\mathbf{z}}$, with $f_{\boldsymbol{\epsilon}}^n(\mathbf{s})$ given by (37), we obtain

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}} \gamma^n(s, z, \mathbf{s}, \tilde{\mathbf{z}}) dz d\tilde{\mathbf{z}} = g^n(s) P(\boldsymbol{\epsilon} = \boldsymbol{\epsilon}^s | S_n = s) f_{\boldsymbol{\epsilon}^s}^n(\mathbf{s}) = P(\boldsymbol{\epsilon} = \boldsymbol{\epsilon}^s) f_{\boldsymbol{\epsilon}^s}^n(\mathbf{s}) = P(\boldsymbol{\epsilon} = \boldsymbol{\epsilon}^s) \frac{1}{|\mathcal{A}_{\boldsymbol{\epsilon}^s}^n|}.$$

The first term is the likelihood that the independently generated increments corresponding to those of \mathbf{s} , while the second term is the chance that these increments will be arranged by the uniform permutation in an order that produces \mathbf{s} . Hence, the marginal correspond to the distribution of \mathbf{S} .

It only remains to show that the pair (\mathbf{S}, \mathbf{Z}) satisfies the bounds (64) and (65). Note that for $1 \leq i \leq n$, recalling (70), we have

$$\begin{aligned} |S_i - Z_i| &= \left| W_i + \frac{i}{n} S - \left(\tilde{Z}_i + \frac{i}{n} Z \right) \right| \\ &\leq |W_i - \tilde{Z}_i| + \frac{i}{n} |S - Z|. \end{aligned} \quad (72)$$

From (69), one can easily check that the conditional distribution of $(\mathbf{S}, \tilde{\mathbf{Z}})$ given $(\boldsymbol{\epsilon}^S, Z) = (\boldsymbol{\epsilon}, z)$ is $\rho_{\boldsymbol{\epsilon}}^n(\mathbf{s}, \tilde{\mathbf{z}})$.

Let $\gamma^2 = n^{-1} \sum_{i=1}^n \epsilon_i^2$ and recall $\nu = \min_{a \in \mathcal{A}} |a| > 0$. As $\gamma \geq \nu$ and $4\lambda \leq \lambda_0$ by (66), we may invoke Theorem 3.1 conditional on $\{\boldsymbol{\epsilon}, Z\}$, and choosing $\eta = \gamma$ we obtain

$$\mathbb{E}(\exp(4\lambda \max_{1 \leq i \leq n} |W_i - \gamma \tilde{Z}_i|) | \boldsymbol{\epsilon}, Z) \leq \exp\left(C \log n + \frac{16K_1 \lambda^2 S_n^2}{n}\right), \quad (73)$$

with C and K_1 depending only on \mathcal{A} . Applying the Cauchy-Schwarz inequality and (68), as S and Z are measurable with respect to $\{\boldsymbol{\epsilon}, Z\}$, from (72) we obtain

$$\begin{aligned} &\mathbb{E} \exp(\lambda \max_{1 \leq i \leq n} |S_i - Z_i|) \\ &\leq \left[\mathbb{E} \left(\mathbb{E}(\exp(\lambda \max_{1 \leq i \leq n} |W_i - \tilde{Z}_i|) | \boldsymbol{\epsilon}, Z) \right)^2 \mathbb{E} \exp(2\lambda |S - Z|) \right]^{1/2} \\ &\leq \left[8 \mathbb{E} \left(\mathbb{E}(\exp(\lambda \max_{1 \leq i \leq n} |W_i - \tilde{Z}_i|) | \boldsymbol{\epsilon}, Z) \right)^2 \right]^{1/2}. \end{aligned} \quad (74)$$

Using conditional Jensen's inequality, the triangle inequality and the convexity of the exponential function in the first three lines below, (73) yields

$$\begin{aligned} &\left(\mathbb{E}(\exp(\lambda \max_{1 \leq i \leq n} |W_i - \tilde{Z}_i|) | \boldsymbol{\epsilon}, Z) \right)^2 \\ &\leq \mathbb{E}(\exp(2\lambda \max_{1 \leq i \leq n} |W_i - \tilde{Z}_i|) | \boldsymbol{\epsilon}, Z) \\ &\leq \frac{1}{2} \mathbb{E}(\exp(4\lambda \max_{1 \leq i \leq n} |W_i - \gamma \tilde{Z}_i|) | \boldsymbol{\epsilon}, Z) + \frac{1}{2} \mathbb{E}(\exp(4\lambda \max_{1 \leq i \leq n} |\gamma \tilde{Z}_i - \tilde{Z}_i|) | \boldsymbol{\epsilon}, Z) \\ &\leq \frac{1}{2} \exp\left(C \log n + \frac{16K_1 \lambda^2 S_n^2}{n}\right) + \frac{1}{2} \mathbb{E}(\exp(4\lambda |\gamma - 1| \max_{1 \leq i \leq n} |\tilde{Z}_i|) | \boldsymbol{\epsilon}, Z) \\ &\leq \exp(C \log n) + \frac{1}{2} \mathbb{E}(\exp(4\lambda |\gamma - 1| \max_{1 \leq i \leq n} |\tilde{Z}_i|) | \boldsymbol{\epsilon}, Z). \end{aligned} \quad (75)$$

For the first term in the fourth line, Lemma 2.6 yields

$$\mathbb{E} \exp \left(\frac{16K_1 \lambda^2 S_n^2}{n} \right) \leq 2,$$

since ε_1 has mean zero, $|\varepsilon_1| \leq B$ in (17) and $4\sqrt{K_1}\lambda \leq \vartheta_{\ell(\varepsilon)}$ by (66).

For the second term in (75), observe that conditional on $(\boldsymbol{\varepsilon}, Z)$, $\tilde{\mathbf{Z}}$ is a mean zero multivariate Gaussian random vector with covariance given by (38). Equivalently, conditional on $(\boldsymbol{\varepsilon}, Z)$, the distribution of $(\tilde{Z}_i/\sqrt{n})_{1 \leq i \leq n}$ is that of a Brownian bridge on $[0, 1]$ sampled at times $1/n, 2/n, \dots, 1$. Thus, letting $B_t, t \in [0, 1]$ be a Brownian bridge independent of $(\boldsymbol{\varepsilon}, Z)$, since γ is a function of $\boldsymbol{\varepsilon}$, we have

$$\begin{aligned} & \mathbb{E}(\exp(4\lambda|\gamma-1| \max_{1 \leq i \leq n} |\tilde{Z}_i|) | \boldsymbol{\varepsilon}, Z) \\ &= \mathbb{E}(\exp(4\sqrt{n}\lambda|\gamma-1| \max_{1 \leq i \leq n} \frac{|\tilde{Z}_i|}{\sqrt{n}}) | \boldsymbol{\varepsilon}, Z) \\ &= \mathbb{E}(\exp(4\sqrt{n}\lambda|\gamma-1| \max_{t \in [n]/n} |B_t|) | \boldsymbol{\varepsilon}, Z) \\ &\leq \mathbb{E}(\exp(4\sqrt{n}\lambda|\gamma-1| \max_{0 \leq t \leq 1} |B_t|) | \boldsymbol{\varepsilon}, Z) \\ &\leq \mathbb{E}(\exp(4\sqrt{n}\lambda|\gamma-1| \max_{0 \leq t \leq 1} B_t) + \exp(4\sqrt{n}\lambda|\gamma-1| \max_{0 \leq t \leq 1} (-B_t)) | \boldsymbol{\varepsilon}, Z). \end{aligned}$$

From [25], the distribution of $X = \max_{0 \leq t \leq 1} B_t$ is given by

$$P(X \leq x) = 1 - \exp(-2x^2) \quad \text{for } x \geq 0.$$

Using this identity, and the fact that $-B_t$ is also a Brownian bridge, it is straightforward to show that for any real number a , we have

$$\mathbb{E}(\exp(a \max_{0 \leq t \leq 1} B_t) + \exp(a \max_{0 \leq t \leq 1} (-B_t))) \leq 2 + \sqrt{2\pi}a \exp(a^2/8).$$

Thus, since B_t and γ are respectively independent of, and a function of, $\boldsymbol{\varepsilon}$, we obtain

$$\begin{aligned} & \mathbb{E}(\exp(4\lambda|\gamma-1| \max_{1 \leq i \leq n} |\tilde{Z}_i|) | \boldsymbol{\varepsilon}, Z) \\ &\leq 2 + \sqrt{2\pi}4\sqrt{n}\lambda|\gamma-1| \exp(2\lambda^2n(\gamma-1)^2) \\ &\leq 2 + 4(B+1)\sqrt{2\pi n}\lambda \exp(2\lambda^2n(\gamma^2-1)^2) \end{aligned} \tag{76}$$

where in the last step, we used $|\gamma-1| \leq B+1$ where B is given by (17), and that $\gamma \geq 0$ implies $1 \leq (\gamma+1)^2$.

Since $\mathbb{E}\varepsilon_1^2 = 1$, we have $n(\gamma^2-1)^2 = (\sum_{i=1}^n (\varepsilon_i^2 - \mathbb{E}\varepsilon_i^2))^2/n$ and $\mathbb{E}(\varepsilon_i^2 - \mathbb{E}\varepsilon_i^2) = 0$. As $\varepsilon^2 \leq B^2$ and $0 \leq \sqrt{2}\lambda \leq \vartheta_{\ell(\varepsilon^2)}$, by (66), Lemma 2.6 yields

$$\mathbb{E} \exp(2\lambda^2n(\gamma^2-1)^2) \leq 2.$$

Additionally, since $\lambda(B+1) \leq 1$ by (66), taking expectation in (76) yields

$$\mathbb{E}(\exp(4\lambda|\gamma-1| \max_{1 \leq i \leq n} |\tilde{Z}_i|)) = 2 + 8(B+1)\sqrt{2\pi n}\lambda \leq \exp(C_1 \log n) \tag{77}$$

for some universal constant C_1 .

Thus, by (74), (75) and (77), we have

$$\begin{aligned}
& \mathbb{E} \exp(\lambda \max_{1 \leq i \leq n} |S_i - Z_i|) \\
& \leq \left[8 \mathbb{E} \left(\exp(C \log n) + \frac{1}{2} \mathbb{E}(\exp(4\lambda |\gamma - 1| \max_{1 \leq i \leq n} |\tilde{Z}_i|) | \boldsymbol{\epsilon}, Z) \right) \right]^{1/2} \\
& \leq 8^{1/2} \left[\exp(C \log n) + \frac{1}{2} \exp(C_1 \log n) \right]^{1/2} \\
& \leq A \exp(A \log n)
\end{aligned}$$

for some universal constant A , which we may take to be at least 8. The proof of (65) is now complete. Lastly note that $\tilde{Z}_n = 0$ implies $Z_n = Z$, hence (68) yields (64) as $A \geq 8$. \square

Theorem 1.3 follows from Lemma 4.1 in exactly the same way as Theorem 1.5 follows from Lemma 5.1 in [6], noting that the reasoning applied at this step does not depend on the support of the summand variables of the random walk.

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